Integration involving powers of Trigonometrical Function :

(1) If the integral of the form:

$$\int \sin^n x\, dx \quad \text{or} \quad \int \cos^m x\, dx$$

(a) when n and m are odd :

$$\int \sin^n x\, dx$$

$$= \int \underbrace{\sin^{n-1} x}_{\text{even}} \cdot \sin x\, dx$$

$$= \int (\sin^2 x)^{\frac{n-1}{2}} \cdot \sin x\, dx$$

putting $\sin^2 x = 1 - \cos^2 x$
and substitute $\cos x = t$
then $\int$, stop

$$\int \cos^m x\, dx$$

$$\int \cos^{m-1} x \cos x\, dx$$

$$\int (\cos^2 x)^{\frac{m-1}{2}} \cos x\, dx$$

putting $\cos^2 x = 1 - \sin^2 x$
and substitute $\sin x = t$
then $\int$ stop.

Ex (1) $\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx$

$$= \int (1 - \cos^2 x) \sin x\, dx$$

$$= \int (1 - t^2) \sin x \frac{dt}{-\sin x}$$

putting $\cos x = t$
$\Rightarrow -\sin x = \frac{dt}{dx}$
$\Rightarrow dx = \frac{dt}{-\sin x}$

$$= -\int (1 - t^2)\, dt$$

$$= -\int dt + \int t^2\, dt$$

$$= -t + \frac{t^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{x} + C$$

(2). $\int \cos^7 x\, dx = \int \cos^6 x \cdot \cos x\, dx$

$$= \int (\cos^2 x)^3 \cos x\, dx$$

$$= \int (1 - \sin^2 x)^3 \cos x\, dx$$

$$= \int (1 - t^2)^3 \cos x \frac{dt}{\cos x}$$

Substitute $\sin x = t$
$\Rightarrow \cos x = \frac{dt}{dx}$
$\Rightarrow dx = \frac{dt}{\cos x}$

$$= \int (1 - t^2)^3\, dt$$

$$= \int (1 - 3t^2 + 3t^4 - t^6)\, dt$$

$$= \int dt - \int 3t^2\, dt + \int 3t^4\, dt - \int t^6\, dt$$

$$= \int dt - 3\int t^2\, dt + 3\int t^4\, dt - \int t^6\, dt$$

$$= t - 3\frac{t^3}{3} + 3\frac{t^5}{5} - \frac{t^7}{7} + C$$

$$= \sin x - 3\frac{\sin^3 x}{3} + 3\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

(b) when n and m are even:

$\int \sin^n x \, dx$ or $\int \cos^m x \, dx$

$= \int (\sin^2 x)^{\frac{n}{2}} dx$ $= \int (\cos^2 x)^{\frac{m}{2}} dx$

(i) putting $\sin^2 x = \frac{1-\cos 2x}{2}$ or (i) putting $\cos^2 x = \frac{1+\cos 2x}{2}$

(ii) · Look your problem it may reduce odd or even power of sine or cosine · Then follow the odd or even principle, then stop. — same

Ex: (1) $\int \sin^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right]$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

(2) $\int \cos^4 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$= \int \left(\frac{1 + \cos^2 2x + 2\cos 2x}{4}\right) dx$

$= \frac{1}{4}\left[\int dx + \int \cos^2 2x \, dx + \int 2\cos 2x \, dx\right]$

$= \frac{1}{4}\left[x + \int \left(\frac{1+\cos 4x}{2}\right) dx + 2\int \cos 2x \, dx\right]$

$= \frac{1}{4}\left[x + \frac{1}{2}\left[\int dx + \int \cos 4x \, dx\right] + 2\frac{\sin 2x}{2}\right]$

$= \frac{1}{4}\left[x + \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) + \sin 2x\right] + C$

$= \frac{1}{4}\left[x + \frac{1}{2}x + \frac{\sin 4x}{8} + \sin 2x\right] + C$

$= \frac{1}{4}\left[\frac{3}{2}x + \frac{\sin 4x}{8} + \sin 2x\right] + C$

(c) If the integral of the form:

$$\int \sin^n x \cdot \cos^m x \, dx$$

Possibilities of n and m

	n	m	
(i)	odd	odd	→ follow the odd principle for small n or m
(ii)	odd	even	→ follow the odd principle for small n or m
(iii)	even	odd	→ follow the odd principle for small n or m
(iv)	even	even	→ follow the even principle

Both or → both equal → Applying trigonometrical formulas

Ex:- (1) $\int \cos^5 x \sin^3 x \, dx$

$= \int \cos^5 x \sin^2 x \cdot \sin x \, dx$

$= \int \cos^5 x (1-\cos^2 x) \sin x \, dx$

$= \int t^5 (1-t^2) \sin x \frac{dt}{-\sin x}$

$= -\int (t^5 - t^7) \, dt$

$= - \int t^5 dt + \int t^7 dt$

$= - \frac{t^6}{6} + \frac{t^8}{8} + C$

$= - \frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$

5 3 small

Substitute $\cos x = t$

$\Rightarrow -\sin x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{-\sin x}$

(2) $\int \sin^5 x \cos^{11} x \, dx$

$= \int \sin^4 x \cdot \sin x \cdot \cos^{11} x \, dx$

$= \int (\sin^2 x)^2 \cdot \sin x \cos^{11} x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \cdot \cos^{11} x \, dx$

$= \int (1-t^2)^2 \sin x \cdot t^{11} \frac{dt}{-\sin x}$

$= -\int (1+t^4-2t^2) t^{11} \, dt$

$= -\int (t^{11} + t^{15} - 2t^{13}) dt$

$= -\int t^{11} dt - \int t^{15} dt + 2\int t^{13} dt$

$= - \frac{t^{12}}{12} - \frac{t^{16}}{16} + 2 \frac{t^{14}}{14} + C$

$= - \frac{\cos^{12} x}{12} - \frac{\cos^{16} x}{16} + \frac{\cos^{14} x}{7} + C$

Substitute $\cos x = t$

$\Rightarrow -\sin x = \frac{dt}{dx}$

$\Rightarrow dt = \frac{dx}{-\sin x}$

(2) If the integrand of the form:

$\int \sec^n x\,dx$ or $\int \csc^m x\,dx$

$\Longrightarrow n, m$ both even $\Longleftarrow$

$= \int \sec^{n-2}x \sec^2 x\,dx$. $= \int \csc^{m-2}x \csc^2 x\,dx$

$= \int (\sec^2 x)^{\frac{n-2}{2}} \sec^2 x\,dx$ $= \int (\csc^2 x)^{\frac{m-2}{2}} \cdot \csc^2 x\,dx$

$= \int (1+\tan^2 x)^{\frac{n-2}{2}} \sec^2 x\,dx$ $= \int (1+\cot^2 x)^{\frac{m-2}{2}} \csc^2 x\,dx$

Substitute $\tan x = t$ Substitute $\cot x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$ and then $\int$

and then $\int$ step. Step.

Ex :- (1) $\int \sec^6 x\,dx = \int \sec^4 x \cdot \sec^2 x\,dx$

$= \int (\sec^2 x)^2 \sec^2 x\,dx$

$= \int (1+\tan^2 x)^2 \sec^2 x\,dx$

$= \int (1+t^2)^2 \sec^2 x \frac{dt}{\sec^2 x}$

$= \int (1+t^2)^2\,dt$

$= \int (1+t^4+2t^2)\,dt$

$= \int dt + \int t^4\,dt + 2\int t^2\,dt$

$= t + \frac{t^5}{5} + 2\frac{t^3}{3} + C$

$= \tan x + \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + C$

Substitute $\tan x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

(2) $\int \csc^4 x\,dx = \int \csc^2 x \csc^2 x\,dx$

$= \int (1+\cot^2 x) \csc^2 x\,dx$

$= \int (1+t^2) \csc^2 x \frac{dt}{-\csc^2 x}$

$= -\int (1+t^2)\,dt$

$= -\int dt - \int t^2\,dt$

$= -t - \frac{t^3}{3} + C$

$= -\cot x - \frac{\cot^3 x}{3} + C$

Substitute $\cot x = t$

$\Rightarrow -\csc^2 x = \frac{dt}{dx}$

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* $\int \sec^n x\,dx$ or $\int \csc^m x\,dx$

$\Rightarrow$ n, m both odd

$\Downarrow$

Follow reduction formulae or integration by parts (later to be taught)

(3) If the integrand of the form:

$\int \tan^n x\,dx$ or $\int \cot^n x\,dx$

$\Rightarrow$ n odd or even

n even: $\int \tan^n x\,dx$

$= \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

Putting $\tan x = t$ $\Rightarrow \int$ stop; Repeat the same procedure

n odd

$\int \tan^n x\,dx$

$= \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x (\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

putting $\tan x = t$ $\Rightarrow \int$ stop. Repeat

(2) $\int \cot^7 x\,dx = \int \cot^5 x \cot^2 x\,dx$

$= \int \cot^5 x (\csc^2 x - 1)\,dx$

$= \int \cot^5 x \csc^2 x\,dx - \int \cot^5 x\,dx$

$= I_1 - I_2$ (say)

$I_1 = \int \cot^5 x \csc^2 x\,dx$

Substitute $\cot x = t$

$\Rightarrow -\csc^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{-\csc^2 x}$

$= \int t^5 \csc^2 x \frac{dt}{-\csc^2 x}$

$= -\int t^5\,dt$

$= -\frac{t^6}{6} + C = -\frac{\cot^6 x}{6} + C_1$

$I_2 = \int \cot^5 x\,dx = \int \cot^3 x \cot^2 x\,dx$

$= \int \cot^3 x (\csc^2 x - 1)\,dx$

$= \int \cot^3 x \csc^2 x\,dx - \int \cot^3 x\,dx$

$= \int \cot^3 x \csc^2 x\,dx - \int \cot x \cot^2 x\,dx$

$= \int \cot^3 x \csc^2 x\,dx - \int \cot x (\csc^2 x - 1)\,dx$

$= \int \cot^3 x \csc^2 x\,dx - \int \cot x \csc^2 x\,dx + \int \cot x\,dx$

$$= \int t^3 \csc^2 x \frac{dt}{-\csc^2 x} - \int t \csc^2 x \frac{dt}{-\csc^2 x} + \log \sin x$$

Substitute $\cot x = t$

$\Rightarrow -\csc^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{-\csc^2 x}$

$$= -\int t^3 dt + \int t\, dt + \log \sin x$$

$$= -\frac{t^4}{4} + \frac{t^2}{2} + \log \sin x + C_2$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log \sin x + C_2$$

$$\therefore \int \cot^7 x\, dx = I_1 - I_2$$

$$= -\frac{\cot^6 x}{6} + C_1 - \left[-\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log \sin x + C_2\right]$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log \sin x + \boxed{C_1 - C_2}$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log \sin x + C$$

(4) If the integrand of the form:

$$\int \sec^n x \cdot \tan^m x\, dx \quad \text{or} \quad \int \csc^n x \cdot \cot^m x\, dx$$

n, m both odd

$$= \int \sec^{n-1} x \cdot \tan^{m-1} x \sec x \cdot \tan x\, dx$$

putting $\tan^2 x = \sec^2 x - 1$

Substitute $\sec x = t$

and $\int$ stop

Ex:- (1) $\int \sec^5 x \cdot \tan^3 x\, dx$

$$= \int \sec^4 x \cdot \sec x \cdot \tan^2 x \cdot \tan x\, dx$$

$$= \int \sec^4 x \cdot \tan^2 x \sec x \cdot \tan x\, dx$$

$$= \int \sec^4 x \cdot (\sec^2 x - 1) \sec x \cdot \tan x\, dx$$

$$= \int t^4 (t^2 - 1) \sec x \tan x \frac{dt}{\sec x \tan x}$$

Substitute $\sec x = t$

$\Rightarrow \sec x \tan x = \frac{dt}{dx}$

$$= \int t^4 (t^2 - 1) dt$$

$$= \int t^6 dt - \int t^4 dt$$

$$= \frac{t^7}{7} - \frac{t^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

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$\int \sec^n x \tan^m x\,dx$ or $\int \csc^n x \cdot \cot^m x\,dx$

n, m both even

$= \int \sec^{n-2} x \cdot \sec^2 x \tan^m x\,dx$

$= \int (\sec^2 x)^{\frac{n-2}{2}} \sec^2 x \cdot \tan^m x\,dx$

$= \int (1+\tan^2 x)^{\frac{n-2}{2}} \cdot \sec^2 x \cdot \tan^m x\,dx$

Putting $\tan x = t$

and $\int$ stop

① $\int \sec^6 x \cdot \tan^8 x\,dx = \int \sec^4 x \cdot \tan^8 x \cdot \sec^2 x\,dx$

$= \int (\sec^2 x)^2 \cdot \tan^8 x \cdot \sec^2 x\,dx$

$= \int (1+\tan^2 x)^2 \cdot \tan^8 x \cdot \sec^2 x\,dx$

$= \int (1+t^2)^2 \cdot t^8 \cdot \sec^2 x \frac{dt}{\sec^2 x}$

$= \int (1+t^2)^2\, t^8\,dt$

$= \int (1+t^4+2t^2)\, t^8\,dt$

$= \int (t^8 + t^{12} + 2t^{10})\,dt$

$= \int t^8\,dt + \int t^{12}\,dt + 2\int t^{10}\,dt$

$= \frac{t^9}{9} + \frac{t^{13}}{13} + 2\frac{t^{11}}{11} + C$

$= \frac{\tan^9 x}{9} + \frac{\tan^{13} x}{13} + 2\frac{\tan^{11} x}{11} + C$

Substitute $\tan x = t$

$\Rightarrow \sec^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

② $\int \csc^4 x \cot^4 x\,dx = \int \csc^2 x \cdot \csc^2 x \cdot \cot^4 x\,dx$

$= \int (1+\cot^2 x) \cdot \csc^2 x \cdot \cot^4 x\,dx$

$= \int (1+t^2)\, \csc^2 x \cdot t^4 \frac{dt}{-\csc^2 x}$

$= -\int (1+t^2)\, t^4\,dt$

$= -\int (t^4 + t^6)\,dt$

$= -\int t^4\,dt - \int t^6\,dt$

$= -\frac{t^5}{5} - \frac{t^7}{7} + C$

$= -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C$

Substitute $\cot x = t$

$\Rightarrow -\csc^2 x = \frac{dt}{dx}$

$\Rightarrow dx = \frac{dt}{-\csc^2 x}$

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* $\int \sec^n x \cdot \tan^m x\, dx$ ⟶ n odd, m even ⟵ $\int \operatorname{cosec}^n x \cdot \cot^m x\, dx$

⇓

follow integration by parts (Later)

* $\int \sec^n x \cdot \tan^m x\, dx$ ⟶ n even, m odd ⟵ $\int \operatorname{cosec}^n x \cdot \cot^m x\, dx$

$= \int \sec^{n-2} x \cdot \tan^m x \cdot \sec^2 x\, dx$

$= \int (\sec^2 x)^{\frac{n-2}{2}} \tan^m x \cdot \sec^2 x\, dx$

$= \int (1+\tan^2 x)^{\frac{n-2}{2}} \cdot \tan^m x \cdot \sec^2 x\, dx$

Substitute $\tan x = t$ & $\int$ stop.

Similarly $\sec x \to \operatorname{cosec} x$, $\tan x \to \cot x$.

Ex:

(1) $\int \sec^6 x \cdot \tan^5 x\, dx = \int \sec^4 x \cdot \sec^2 x \cdot \tan^5 x\, dx$

$= \int (\sec^2 x)^2 \cdot \sec^2 x \cdot \tan^5 x\, dx$

$= \int (1+\tan^2 x)^2 \cdot \sec^2 x \cdot \tan^5 x\, dx$

Substitute $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx}$

$= \int (1+t^2)^2 \cancel{\sec^2 x} \cdot t^5 \frac{dt}{\cancel{\sec^2 x}}$

$= \int (1+t^4+2t^2)\, t^5\, dt$

$= \int (t^5+t^9+2t^7)\, dt$

$= \int t^5 dt + \int t^9 dt + 2\int t^7 dt$

$= \frac{t^6}{6} + \frac{t^{10}}{10} + 2\frac{t^8}{8_4} + C$

$= \frac{\tan^6 x}{6} + \frac{\tan^{10} x}{10} + \frac{\tan^8 x}{4} + C$.

(2) $\int \operatorname{cosec}^4 x \cdot \cot^9 x\, dx = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec}^2 x \cdot \cot^9 x\, dx$

$= \int (1+\cot^2 x)\, \operatorname{cosec}^2 x \cdot \cot^9 x\, dx$

$= \int (1+t^2)\, \cancel{\operatorname{cosec}^2 x} \cdot \cot^9 x \cdot \frac{dt}{-\cancel{\operatorname{cosec}^2 x}}$

Substitute $\cot x = t \Rightarrow -\operatorname{cosec}^2 x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{-\operatorname{cosec}^2 x}$

$= -\int (1+t^2)\, t^9\, dt$

$= -\int t^9 dt - \int t^{11} dt$

$= -\frac{t^{10}}{10} - \frac{t^{12}}{12} + C$

$= -\frac{\cot^{10} x}{10} - \frac{\cot^{12} x}{12} + C$

Integration by Trigonometric Substitution

Form of integrand	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
a^2+x^2	$x = a\tan\theta$ or $x = a\cot\theta$
x^2-a^2	$x = a\sec\theta$ or $x = a\,\mathrm{cosec}\,\theta$

① Integrals of $\frac{1}{a^2+x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$ and $\frac{1}{x\sqrt{x^2-a^2}}$

(a) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Proof: $\int \frac{dx}{x^2+a^2}$. Substitute $x = a\tan\theta$

$\Rightarrow dx = a\sec^2\theta\, d\theta$

$= \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$= \int \frac{a\sec^2\theta}{a^2(\tan^2\theta+1)}\, d\theta$

As $x = a\tan\theta$
$\Rightarrow \tan\theta = x/a$
$\Rightarrow \theta = \tan^{-1} x/a$

$= \int \frac{\cancel{a}\,\cancel{\sec^2\theta}}{a^2\,\cancel{\sec^2\theta}}\, d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(b) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$.

Proof! $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

Substitute $x = a\sin\theta$
$\Rightarrow dx = a\cos\theta\, d\theta$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}}$

$= \int \frac{\cancel{a\cos\theta}\, d\theta}{\cancel{a\cos\theta}} = \int d\theta = \theta + C$

$= \sin^{-1} x/a + C$

As $x = a\sin\theta$
$\Rightarrow \frac{x}{a} = \sin\theta$
$\Rightarrow \theta = \sin^{-1}\frac{x}{a}$

$\Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

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Ex :- $\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{dx}{\sqrt{x^2+4^2}} = \log|x+\sqrt{x^2+4^2}| + C$

OR :- $\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{dx}{\sqrt{x^2+4^2}}$ Substitute $x = 4\tan\theta$
$\Rightarrow dx = 4\sec^2\theta\, d\theta$

$= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{4^2\tan^2\theta + 4^2}}$

$= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{4^2(\tan^2\theta+1)}}$

$= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{4^2\sec^2\theta}}$

$= \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta$

$\tan\theta = x/4$
$\Rightarrow \sec\theta = \sqrt{1+\tan^2\theta}$
$= \sqrt{1+\frac{x^2}{4^2}}$
$= \frac{\sqrt{x^2+4^2}}{4}$

$= \log|\sec\theta + \tan\theta| + C_1$

$= \log\left|\frac{\sqrt{x^2+4^2}}{4} + \frac{x}{4}\right| + C_1$

$= \log|x+\sqrt{x^2+4^2}| + C_1 - \log 4$

$= \log|x+\sqrt{x^2+16}| + C$

(b) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C$

Proof: LHS $= \int \frac{dx}{\sqrt{x^2-a^2}}$ Substitute $x = a\sec\theta$
$\Rightarrow dx = a\sec\theta.\tan\theta\, d\theta$

$= \int \frac{a\sec\theta.\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{a\sec\theta.\tan\theta\, d\theta}{\sqrt{a^2(\sec^2\theta-1)}}$

$= \int \frac{a\sec\theta.\tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}}$

$= \int \frac{a\sec\theta.\cancel{\tan\theta}\, d\theta}{a\cancel{\tan\theta}}$

As $x = a\sec\theta$
$\Rightarrow x/a = \sec\theta$
$\tan\theta = \sqrt{\sec^2\theta - 1}$
$= \sqrt{\frac{x^2}{a^2} - 1}$
$= \sqrt{\frac{x^2-a^2}{a^2}}$

$= \int \sec\theta\, d\theta$

$= \log|\sec\theta + \tan\theta| + C_1$

$= \log\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1$

$= \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C_1$

$= \log|x+\sqrt{x^2-a^2}| - \log a + C_1$

$= \log|x+\sqrt{x^2-a^2}| + C =$ RHS as $C_1 - \log a = C$ constant

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(3) $\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{x\sqrt{x^2-2^2}} = \frac{1}{2}\sec^{-1}\frac{x}{2} + C$

OR $\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{x\sqrt{x^2-2^2}}$

[Substitute $x = 2\sec\theta$ $\Rightarrow dx = 2\sec\theta\tan\theta\, d\theta$]

$= \int \frac{2\sec\theta\cdot\tan\theta\, d\theta}{2\sec\theta\sqrt{2^2\sec^2\theta - 2^2}}$

$= \int \frac{2\sec\theta\cdot\tan\theta\, d\theta}{2\sec\theta\sqrt{2^2(\sec^2\theta-1)}} = \int \frac{2\sec\theta\cdot\tan\theta\, d\theta}{2\sec\theta\sqrt{2^2\tan^2\theta}}$

$= \int \frac{\cancel{2\sec\theta}\cdot\cancel{\tan\theta}\, d\theta}{\cancel{2\sec\theta}\cdot 2\cancel{\tan\theta}}$

[As $x = 2\sec\theta$ $\Rightarrow \frac{x}{2} = \sec\theta$ $\Rightarrow \theta = \sec^{-1}\frac{x}{2}$]

$= \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}\frac{x}{2} + C$

(2) Integrals of $\frac{1}{\sqrt{x^2+a^2}}$ and $\frac{1}{\sqrt{x^2-a^2}}$

(a) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(b) Proof: LHS $= \int \frac{dx}{\sqrt{x^2+a^2}}$ Substitute $x = a\tan\theta$ $\Rightarrow dx = a\sec^2\theta\, d\theta$

$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$

$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2(\tan^2\theta+1)}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\sec^2\theta}}$

$= \int \frac{\cancel{a}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{a}\cancel{\sec\theta}} = \int \sec\theta\, d\theta$

$= \log\left|\sec\theta + \tan\theta\right| + C_1$

$= \log\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$

$= \log\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C_1$

$= \log\left|\sqrt{x^2+a^2}+x\right| - \log a + C_1$

$= \log\left|x+\sqrt{x^2+a^2}\right| + C$

[As $x = a\tan\theta$ $\Rightarrow \frac{x}{a} = \tan\theta$ $\Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{a}\right)^2} = \sqrt{\frac{a^2+x^2}{a^2}} = \frac{\sqrt{a^2+x^2}}{a}$]

as $C = C_1 - \log a$ = constant

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

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(c) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

$\underline{LHS} = \int \frac{dx}{x\sqrt{x^2-a^2}}$

Substitute $x = a\sec\theta$
$\Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$= \int \frac{a\sec\theta \cdot \tan\theta\, d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{\cancel{a\sec\theta} \cdot \tan\theta\, d\theta}{\cancel{a\sec\theta}\sqrt{a^2(\sec^2\theta-1)}}$

$= \int \frac{\tan\theta\, d\theta}{\sqrt{a^2\tan^2\theta}} = \int \frac{\cancel{\tan\theta}\, d\theta}{a\cancel{\tan\theta}}$

$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$

$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

As $x = a\sec\theta$
$\Rightarrow \frac{x}{a} = \sec\theta$
$\Rightarrow \theta = \sec^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

Problem:

(1) $\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2}$

$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

$\underline{OR}$ $\int \frac{dx}{x^2+9} = \int \frac{dx}{x^2+3^2}$

Substitute $x = 3\tan\theta$
$\Rightarrow dx = 3\sec^2\theta\, d\theta$

$= \int \frac{3\sec^2\theta\, d\theta}{3^2\tan^2\theta + 3^2}$

$= \frac{3\sec^2\theta\, d\theta}{3^2(\tan^2\theta+1)}$

$= \frac{\cancel{3}\cancel{\sec^2\theta}\, d\theta}{3^{\cancel{2}}\cancel{\sec^2\theta}} = \frac{1}{3}\, d\theta$

As $x = 3\tan\theta$
$\Rightarrow \frac{x}{3} = \tan\theta$
$\Rightarrow \theta = \tan^{-1}\frac{x}{3}$

$= \frac{1}{3}\theta + C = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

(2) $\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{4^2-x^2}} = \sin^{-1}\frac{x}{4} + C$

Substitute $x = 4\sin\theta$
$\Rightarrow dx = 4\cos\theta\, d\theta$

$\underline{OR}$ $\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{4^2-x^2}}$

$= \int \frac{4\cos\theta\, d\theta}{\sqrt{4^2-4^2\sin^2\theta}} = \int \frac{4\cos\theta\, d\theta}{\sqrt{4^2(1-\sin^2\theta)}}$

$= \int \frac{4\cos\theta\, d\theta}{\sqrt{4^2\cos^2\theta}} = \int \frac{\cancel{4\cos\theta}}{\cancel{4\cos\theta}}\, d\theta = \int d\theta = \theta + C$

$= \sin^{-1}\frac{x}{4} + C$

Ex:- $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}} = \log|x+\sqrt{x^2-9}| + C$.

OR. $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}}$ Substitute: $x = 3\sec\theta$
$\Rightarrow dx = 3\sec\theta . \tan\theta d\theta$

$= \int \frac{3\sec\theta . \tan\theta d\theta}{\sqrt{3^2\sec^2\theta - 3^2}}$

$= \int \frac{3\sec\theta . \tan\theta d\theta}{\sqrt{3^2(\sec^2\theta - 1)}}$

$= \int \frac{3\sec\theta . \tan\theta d\theta}{\sqrt{3^2\tan^2\theta}}$

$= \int \frac{\cancel{3}\sec\theta . \cancel{\tan\theta} d\theta}{\cancel{3}\cancel{\tan\theta}} = \int \sec\theta d\theta$

$= \log|\sec\theta + \tan\theta| + C_1$

$= \log\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_1$

$= \log|x+\sqrt{x^2-9}| - \log 3 + C_1$

$= \log|x+\sqrt{x^2-9}| + C$

(3) Integral of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$:

(a) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

LHS $= \int \sqrt{a^2-x^2}\,dx$ | Substitute $x = a\sin\theta$
$\Rightarrow dx = a\cos\theta d\theta$

$= \int \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta d\theta$

$= \int \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta d\theta$

$= \int \sqrt{a^2\cos^2\theta} . a\cos\theta d\theta$

$= \int a\cos\theta . a\cos\theta d\theta$

| as $x = a\sin\theta$
$\Rightarrow \sin\theta = x/a$
$\cos\theta = \sqrt{1-\sin^2\theta}$
$= \sqrt{1 - \frac{x^2}{a^2}}$

$= a^2 \int \cos^2\theta\, d\theta$

$= a^2 \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2}\left[\int d\theta + \int \cos 2\theta d\theta\right]$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{2\sin\theta . \cos\theta}{2}\right]$

$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a} . \sqrt{1-\frac{x^2}{a^2}}\right] + C$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$ = RHS

Ex 2: $\int \sqrt{25-x^2}\,dx = \int \sqrt{5^2-x^2}\,dx$

$= \int \sqrt{5^2 - 5^2\sin^2\theta}\; 5\cos\theta\, d\theta$ Substitute $x = 5\sin\theta$ $\Rightarrow dx = 5\cos\theta\, d\theta$

$= \int \sqrt{5^2(1-\sin^2\theta)}\; 5\cos\theta\, d\theta$

$= \int \sqrt{5^2\cos^2\theta}\; 5\cos\theta\, d\theta$

$= \int 5\cos\theta \cdot 5\cos\theta\, d\theta$

$= 25 \int \cos^2\theta\, d\theta = 25 \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{25}{2}\left[\int d\theta + \int \cos 2\theta\, d\theta\right]$

$= \frac{25}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$= \frac{25}{2}\left[\sin^{-1}\frac{x}{5} + \frac{2\sin\theta\cos\theta}{2}\right] + C$

$= \frac{25}{2}\left[\sin^{-1}\frac{x}{5} + \frac{x}{5}\sqrt{1-\frac{x^2}{5^2}}\right] + C$

$= \frac{25}{2}\left[\sin^{-1}\frac{x}{5} + \frac{x}{5}\,\frac{\sqrt{5^2-x^2}}{5}\right] + C$

$= \frac{25}{2}\sin^{-1}\frac{x}{5} + \frac{\cancel{25}x}{2\times\cancel{25}}\sqrt{5^2-x^2} + C$

$= \frac{25}{2}\sin^{-1}\frac{x}{5} + \frac{x}{2}\sqrt{5^2-x^2} + C$

(b) $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C$

(c) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

This two are proved by integration by parts (later)

(4) Integrals of $\frac{1}{x^2-a^2}$ and $\frac{1}{a^2-x^2}$.

(a) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Proof: LHS $= \int \frac{dx}{x^2-a^2}$ | Substitute $x = a\sec\theta$ $\Rightarrow dx = a\sec\theta\cdot\tan\theta\, d\theta$

$= \int \frac{a\sec\theta\,\tan\theta\, d\theta}{a^2\sec^2\theta - a^2}$

$= \int \frac{a\sec\theta\,\tan\theta\, d\theta}{a^2(\sec^2\theta-1)}$

$= \int \frac{\cancel{a}\sec\theta\cdot\cancel{\tan\theta}\, d\theta}{a^{\cancel{2}}\tan^{\cancel{2}}\theta} = \int \frac{\sec\theta}{a\tan\theta}\, d\theta$

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$= \frac{1}{a}\int \operatorname{cosec}\theta\, d\theta = \frac{1}{a}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C$;

$= \frac{1}{a}\log\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + C$

As $x = a\sec\theta$
$\Rightarrow \sec\theta = x/a$
$\cos\theta = a/x$
$\sin\theta = \sqrt{1-\frac{a^2}{x^2}}$
$= \frac{\sqrt{x^2-a^2}}{x}$
$\Rightarrow \operatorname{cosec}\theta = \frac{x}{\sqrt{x^2-a^2}}$
$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{a}{\sqrt{x^2-a^2}}$

$= \frac{1}{a}\log\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + C$

$= \frac{1}{a}\log\left|\frac{x-a}{\sqrt{(x-a)(x+a)}}\right| + C$

$= \frac{1}{a}\log\left|\frac{\sqrt{x-a}}{\sqrt{x+a}}\right| + C$

$= \frac{1}{a}\log\left|\sqrt{\frac{x-a}{x+a}}\right| + C$

$= \frac{1}{a}\log\left|\left(\frac{x-a}{x+a}\right)^{1/2}\right| + C = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
$=$ RHS

Ex: $\int \frac{dx}{x^2-25} = \int \frac{dx}{x^2-5^2}$

Substitute $x = 5\sec\theta$
$\Rightarrow dx = 5\sec\theta\tan\theta\, d\theta$

$= \int \frac{5\sec\theta\cdot\tan\theta\, d\theta}{5^2\sec^2\theta - 5^2}$

$= \int \frac{5\sec\theta\cdot\tan\theta\, d\theta}{5^2(\sec^2\theta - 1)}$

$= \int \frac{5\sec\theta\cdot\tan\theta\, d\theta}{5^2\tan^2\theta} = \int \frac{\sec\theta}{5\tan\theta}\, d\theta$

$= \frac{1}{5}\int \operatorname{cosec}\theta\, d\theta = \frac{1}{5}\log\left|\operatorname{cosec}\theta - \cot\theta\right| + C$

$= \frac{1}{5}\log\left|\frac{x}{\sqrt{x^2-25}} - \frac{5}{\sqrt{x^2-25}}\right| + C$

$= \frac{1}{5}\log\left|\frac{x-5}{\sqrt{x^2-25}}\right| + C = \frac{1}{5}\log\left|\frac{x-5}{x+5}\right|^{1/2} + C$

$= \frac{1}{10}\log\left|\frac{x-5}{x+5}\right| + C$

OR $\int \frac{dx}{x^2-25} = \frac{dx}{x^2-5^2}$ $(a=5)$

$= \frac{1}{2\times5}\log\left|\frac{x-5}{x+5}\right| + C$

$= \frac{1}{10}\log\left|\frac{x-5}{x+5}\right| + C$

(b) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

Proof: LHS $= \int \frac{dx}{a^2-x^2}$ | Substitute $x = a\sin\theta$ $\Rightarrow dx = a\cos\theta\, d\theta$

$= \int \frac{a\cos\theta\, d\theta}{a^2 - a^2\sin^2\theta}$

$= \int \frac{a\cos\theta\, d\theta}{a^2(1-\sin^2\theta)} = \int \frac{a\cos\theta\, d\theta}{a^2\cos^2\theta}$

$= \int \frac{a\cos\theta\, d\theta}{a^2\cos^2\theta} = \frac{1}{a}\int \sec\theta\, d\theta$

$= \frac{1}{a}\log(\sec\theta + \tan\theta) + c$

$= \frac{1}{a}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + c$ | As $x = a\sin\theta$ $\Rightarrow \sin\theta = x/a$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{a^2}}$

$= \frac{1}{a}\log\left|\frac{1+x/a}{\sqrt{1-x^2/a^2}}\right| + c$

$= \frac{1}{a}\log\left|\frac{\frac{a+x}{a}}{\frac{\sqrt{a^2-x^2}}{a}}\right| + c$

$= \frac{1}{a}\log\left|\frac{a+x}{\sqrt{a^2-x^2}}\right| + c$

$= \frac{1}{a}\log\left|\frac{a+x}{a-x}\right|^{1/2} + c$

$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c =$ RHS

Ex: $\int \frac{dx}{16-x^2} = \int \frac{dx}{4^2-x^2}$ Substitute $x = 4\sin\theta$ $\Rightarrow dx = 4\cos\theta\, d\theta$

$= \int \frac{4\cos\theta\, d\theta}{4^2 - 4^2\sin^2\theta} = \int \frac{4\cos\theta\, d\theta}{4^2(1-\sin^2\theta)}$ | as $\sin\theta = x/4$, $\cos\theta = \sqrt{1-\frac{x^2}{16}}$

$= \int \frac{4\cos\theta\, d\theta}{4^2\cos^2\theta} = \frac{1}{4}\int \sec\theta\, d\theta$

$= \frac{1}{4}\log|\sec\theta + \tan\theta| + c$

$= \frac{1}{4}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| + c = \frac{1}{4}\log\left|\frac{1+x/4}{\sqrt{1-x^2/16}}\right| + c$

$$= \frac{1}{4}\log\left|\frac{\frac{4+x}{4}}{\frac{\sqrt{16-x^2}}{4}}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{4+x}{\sqrt{16-x^2}}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{\sqrt{4+x}}{\sqrt{4-x}}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{4+x}{4-x}\right|^{1/2} + c$$

$$= \frac{1}{2\times 4}\log\left|\frac{4+x}{4-x}\right| + c$$

$$= \frac{1}{8}\log\left|\frac{4+x}{4-x}\right| + c$$

Evaluate:

(1) $\int \frac{dx}{\sqrt{2-4x+x^2}} = \int \frac{dx}{\sqrt{x^2-4x+2}} = \int \frac{dx}{\sqrt{x^2-2\cdot x\cdot 2+2^2-2^2+2}}$

$$= \int \frac{dx}{\sqrt{(x-2)^2-4+2}} = \int \frac{dx}{\sqrt{(x-2)^2-2}}$$

$$= \int \frac{dx}{\sqrt{(x-2)^2-(\sqrt{2})^2}}$$

Substituting $x-2=t$ $\Rightarrow dx=dt$

$$= \int \frac{dt}{\sqrt{t^2-(\sqrt{2})^2}}$$

$$= \log\left|t+\sqrt{t^2-(\sqrt{2})^2}\right| + c$$

$$= \log\left|x-2+\sqrt{(x-2)^2-2}\right| + c$$

$$= \log\left|x-2+\sqrt{x^2-4x+2}\right| + c$$

(2) $\int \frac{dx}{\sqrt{8-2x-x^2}} = \int \frac{dx}{\sqrt{8-(2x+x^2)}} = \int \frac{dx}{\sqrt{8-(x^2+2x)}}$

$$= \int \frac{dx}{\sqrt{8-(x^2+2\cdot x\cdot 1+1^2-1)}}$$

$$= \int \frac{dx}{\sqrt{8-(x+1)^2+1}} = \int \frac{dx}{\sqrt{9-(x+1)^2}}$$

$$= \int \frac{dx}{\sqrt{3^2-(x+1)^2}}$$

Substitute $x+1=t$ $\Rightarrow dx=dt$

$$= \int \frac{dt}{\sqrt{3^2-t^2}}$$

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$= \sin^{-1}\left(\frac{t}{3}\right) + c$

$= \sin^{-1}\left(\frac{x+1}{3}\right) + c$

(3) $\int \frac{\cos x\,dx}{\sin^2 x - 25} = \int \frac{\cos x}{t^2 - 25}\frac{dt}{\cos x}$; [Substitute $\sin x = t$ $\rightarrow \cos x\,dx = dt$]

$= \int \frac{dt}{t^2 - 25} = \int \frac{dt}{t^2 - 5^2}$

$= \frac{1}{2\times 5} \log\left|\frac{t-5}{t+5}\right| + c$

$= \frac{1}{10}\log\left|\frac{\sin x - 5}{\sin x + 5}\right| + c$

(4) $\int \sqrt{x^2+3x}\,dx = \int \sqrt{x^2 + 2\cdot\frac{3}{2}\cdot x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$

$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$ | Substitute $x + 3/2 = t$ $\rightarrow dx = dt$

$= \int \sqrt{t^2 - \left(\frac{3}{2}\right)^2}\,dt$

$= \frac{1}{2} t\sqrt{t^2 - (3/2)^2} - \frac{(3/2)^2}{2}\log\left(\frac{t + \sqrt{t^2 - (3/2)^2}}{3/2}\right) + c$

$= \frac{1}{2}(x+3/2)\sqrt{(x+3/2)^2 - (3/2)^2} - \frac{9}{8}\log\left|(x+3/2) + \sqrt{(x+3/2)^2 - (3/2)^2}\right| + c$

$= \frac{1}{2}\left(\frac{2x+3}{2}\right)\sqrt{x^2+3x} - \frac{9}{8}\log\left|x + \frac{3}{2} + \sqrt{x^2+3x}\right| + c$

$= \frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x}\right| + c$

(A) Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

→ 1st the given integral can be expressed as the sum of two integrals which is easily evaluated i.e choose A and B (constant) such that

$px + q = A\frac{d}{dx}(ax^2+bx+c) + B$

Then find A & B, by equating the coefficients of ~~[illegible]~~ like powers of x then ∫ step

Ex :- $\int \frac{2x+1}{\sqrt{x^2+2x-1}}\,dx$

Let $2x+1 = A\frac{d}{dx}(x^2+2x-1)+B$

$\Rightarrow 2x+1 = A(2x+2)+B = 2Ax+2A+B$

Equating the coefficient of like power of x (x^1, x^0) we have

$2 = 2A \Rightarrow A = 2/2 = 1$

$1 = 2A+B \Rightarrow 1 = 2.1+B \Rightarrow B = 1-2 = -1$

$\therefore A = 1,\ B = -1.$

$\therefore \int \frac{2x+1}{\sqrt{x^2+2x-1}}dx = \int \frac{1.(2x+2)-1}{\sqrt{x^2+2x-1}}dx$

$= \int \frac{2x+2}{\sqrt{x^2+2x-1}}dx - \int \frac{dx}{\sqrt{x^2+2x-1}}$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{dx}{\sqrt{x^2+2.x.1+1-1-1}}$ | Substitute $x^2+2x-1 = t$ $\Rightarrow (2x+2)dx = dt$

$= \int t^{-1/2}dt - \int \frac{dx}{\sqrt{(x+1)^2-2}}$

$= \frac{t^{-1/2+1}}{-1/2+1} - \int \frac{dx}{\sqrt{(x+1)^2-(\sqrt{2})^2}}$

$= \frac{t^{1/2}}{1/2} - \log\left|(x+1)+\sqrt{(x+1)^2-(\sqrt{2})^2}\right| + c$

$= 2(x^2+2x-1)^{1/2} - \log\left|(x+1)+\sqrt{(x+1)^2-(\sqrt{2})^2}\right| + c$

$= 2\sqrt{x^2+2x-1} - \log\left|(x+1)+\sqrt{x^2+2x-1}\right| + c$

*(B) Integrals of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int \sqrt{ax^2+bx+c}\,dx$

→ ~~Expressing $\sqrt{ax^2+bx+c}$~~

→ Expressing ax^2+bx+c in the form of $a[(x+\alpha)^2 \pm \beta^2]$ and then integrating.

*(C) Integrals of the form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

→ 1st the integral can be expressed as the sum of two integrals which is evaluated easily (similarly (A))

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Integration by Parts: (Integral of the product of two function)

Let u, v be two differentiable function of x, then

$$d(u.v) = u\,dv + v\,du$$

$$\int d(uv) = \int (u\,dv + v\,du) = \int u\,dv + \int v\,du$$

$$\Rightarrow uv = \int u\,dv + \int v\,du$$

$$\Rightarrow \int u\,dv = uv - \int v\,du = u\int dv - \int \left(\int dv\right) du$$

$\therefore \int \underset{1st}{u}\, \underset{2nd}{dv}$ = 1st function × integral of 2nd function − Integral of [integral of 2nd function × differentiate of 1st function]

To choice which is 1st and 2nd function of a given integral are follow ILATE Principle.

I = Inverse Trigonometrical Function

L = Logarithmic function

A = Algebraic Function

T = Trigonometric function

E = Exponential function

But this rule sometimes is no effective, depends on problems

$$\int uv\,dx = u\int v\,dx - \int \left[\frac{du}{dx}\int v\,dx\right] dx$$

Ex: $\int \underset{1}{x}\,\underset{2}{\cos x}\,dx = x\int \cos x\,dx - \int \left[\frac{dx}{dx}\int \cos x\,dx\right] dx$

$= x\sin x - \int 1.\sin x\,dx$

$= x\sin x - \int \sin x\,dx = x\sin x - (-\cos x) + C$

$= x\sin x + \cos x + C$

$\int \ln x\,dx = \int \underset{1}{\ln x}.\underset{2}{1}\,dx = \ln x \int 1\,dx - \int \left(\frac{d}{dx}(\ln x) \int 1\,dx\right) dx$

$= \ln x.\,x - \int \frac{1}{x}.x\,dx$

$= x\ln x - \int dx = x\ln x - x + C$

Integral of the form: $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$

LHS $= \int e^x [f(x) + f'(x)]dx = \int \underset{2}{e^x}\,\underset{1}{f(x)}\,dx + \int e^x f'(x)\,dx$

$= f(x)\int e^x dx - \int \left(\frac{d}{dx}f(x).\int e^x dx\right) dx + \int e^x f'(x)\,dx$

$= f(x)\,e^x - \cancel{\int f'(x).\,e^x\,dx} + \cancel{\int e^x f'(x)\,dx}$

$= e^x f(x) + C$ = RHS.

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Ex:- (1) $\int e^x(\sin x+\cos x)\,dx$

Let $f(x)=\sin x \Rightarrow f'(x)=\cos x$

$\therefore \int e^x(f(x)+f'(x))\,dx = e^x f(x)+C$

i.e. $\int e^x(\sin x+\cos x)\,dx = e^x\sin x + C$

OR $\int e^x(\sin x+\cos x)\,dx = \int e^x \sin x\,dx + \int e^x\cos x\,dx$

$= \sin x\int e^x dx - \int\left(\frac{d}{dx}\sin x \cdot \int e^x dx\right)dx + \int e^x\cos x\,dx$

$= \sin x\; e^x - \int \cos x\cdot e^x\,dx + \int e^x\cos x\,dx$

$= e^x\sin x + C$

(2) $\int \frac{e^x}{x}(1+x\log x)\,dx = \int e^x\left(\frac{1}{x}+\log x\right)dx$

Let $f(x)=\log x \Rightarrow f'(x)=\frac{1}{x}$

$\therefore \int e^x\left(\frac{1}{x}+\log x\right)dx = \int e^x(f'(x)+f(x))\,dx$

$= e^x f(x)+C$

$= e^x\log x + C$.

(3) $\int \frac{xe^x}{(x+1)^2}\,dx = \int e^x\left(\frac{x+1-1}{(x+1)^2}\right)dx = \int e^x\left[\frac{x+1}{(x+1)^2}-\frac{1}{(x+1)^2}\right]dx$

$= \int e^x\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$

Let $f(x)=\frac{1}{x+1} \Rightarrow f'(x) = -\frac{1}{(x+1)^2}$

$\therefore \int e^x\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx = \int e^x\, [f(x)+f'(x)]\,dx$

$= e^x f(x)+C = \frac{e^x}{x+1}+C$.

(4) $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int e^x\left(\frac{1+2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$

$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$

Let $f(x)=\tan\frac{x}{2} \Rightarrow f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

$\therefore \int e^x\left(\tan\frac{x}{2}+\frac{1}{2}\sec^2\frac{x}{2}\right)dx = \int e^x(f(x)+f'(x))\,dx$

$= e^x f(x)+C$

$= e^x\tan\frac{x}{2}+C$

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Problems:

(1) $\int \underset{2}{x} \underset{1}{\tan^{-1}x}\, dx = \tan^{-1}x \int x\,dx - \int \left[\frac{d}{dx}(\tan^{-1}x) \int x\,dx\right] dx$

$= \tan^{-1}x \; \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int \left(1 - \frac{1}{x^2+1}\right) dx\right]$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$.

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$.

$= \frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x + C$

(2) $\int \log(1+x^2)\, dx = \int \log(1+x^2) \cdot 1\, dx$

$= \log(1+x^2) \int 1\, dx - \int \left[\frac{d}{dx}\log(1+x^2) \int 1 \cdot dx\right] dx$

$= \log(1+x^2) \cdot x - \int \left(\frac{1}{1+x^2} \cdot 2x \cdot x\right) dx$

$= x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}\, dx$

$= x\log(1+x^2) - 2\int \left[\frac{x^2+1-1}{x^2+1}\right] dx$

$= x\log(1+x^2) - 2\int \left(1 - \frac{1}{x^2+1}\right) dx$

$= x\log(1+x^2) - 2\int dx + 2\int \frac{dx}{x^2+1}$

$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$

(3) $\int \sec^3 x\, dx = \int \underset{1}{\sec x} \cdot \underset{2}{\sec^2 x}\, dx$

$= \sec x \int \sec^2 x\, dx - \int \left[\frac{d}{dx}\sec x \cdot \int \sec^2 x\, dx\right] dx$

$= \sec x \tan x - \int \sec x \cdot \tan x \cdot \tan x\, dx$

$= \sec x \cdot \tan x - \int \sec x \tan^2 x\, dx$

$= \sec x \cdot \tan x - \int \sec x \cdot (\sec^2 x - 1)\, dx$

$= \sec x \cdot \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\Rightarrow \int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x \cdot \tan x + \int \sec x\, dx$

$\Rightarrow 2\int \sec^3 x\, dx = \sec x \cdot \tan x + \log\left|\tan\left(\pi/4 + \frac{x}{2}\right)\right| + C$

$\Rightarrow \int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \cdot \tan x + \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|\right] + C$

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④ $\int e^{ax}\cos bx\,dx = \cos bx\int e^{ax}dx - \int\left[\frac{d}{dx}\cos bx\cdot\int e^{ax}dx\right]dx$

$= \cos bx\,\frac{e^{ax}}{a} - \int(-\sin bx)\cdot b\cdot\frac{e^{ax}}{a}\,dx$

$= \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx\int e^{ax}dx - \int\left(\frac{d}{dx}\sin bx\int e^{ax}dx\right)dx\right]$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx\,\frac{e^{ax}}{a} - \int\cos bx\cdot b\,\frac{e^{ax}}{a}\,dx\right]$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\sin bx\,\frac{e^{ax}}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right]$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx + C$

$\Rightarrow \int e^{ax}\cos bx\,dx + \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}e^{ax}\sin bx + C$

$\Rightarrow \left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2}\left[a\cos bx + b\sin bx\right] + C$

$\Rightarrow \int e^{ax}\cos bx\,dx = \dfrac{\frac{e^{ax}}{a^2}(a\cos bx + b\sin bx) + C}{1+\frac{b^2}{a^2}}$

$= \frac{e^{ax}}{a^2}(a\cos bx + b\sin bx)\times\frac{a^2}{a^2+b^2} + \frac{Ca^2}{a^2+b^2}$

$\Rightarrow \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C_1$

where $C_1 = \frac{Ca^2}{a^2+b^2}$ = const

⑤ $\int(\log x)^2dx = \int(\log x)^2\cdot 1\,dx$

$= (\log x)^2\int 1\,dx - \int\left[\frac{d}{dx}(\log x)^2\cdot\int 1\,dx\right]dx$

$= (\log x)^2\,x - \int 2\log x\cdot\frac{1}{x}\cdot x\,dx$

$= x(\log x)^2 - 2\int\log x\,dx$

$= x(\log x)^2 - 2\left[\log x\cdot\int dx - \int\left(\frac{d}{dx}(\log x)\cdot\int dx\right)dx\right]$

$= x(\log x)^2 - 2\left[\log x\cdot x - \int\frac{1}{x}\cdot x\,dx\right]$

$= x(\log x)^2 - 2\left(x\log x - \int dx\right)$

$= x(\log x)^2 - 2(x\log x - x) + C$

$= x(\log x)^2 - 2x\log x + 2x + C$

Definite Integral

→ An integral expressed as the difference between the values of the integral at specified upper and lower limits of the independent variable.

→ It is denoted by $\int_a^b f(x)\,dx$ where a = lower limit, b = upper limit

and evaluated as

$$\int_a^b f(x)\,dx = \left[F(x)+c\right]_a^b. \quad \left[\because \int f(x)dx = F(x)+c\right]$$

$$= F(b)+c-(F(a)+c)$$

$$= F(b)+c-F(a)+c$$

$$= F(b)-F(a)$$

$$\therefore \int_a^b f(x)\,dx = F(b)-F(a)$$

Ex: $\int_1^2 x^2\,dx = \left[\frac{x^3}{3}+c\right]_1^2 = \frac{2^3}{3}+c-\frac{1^3}{3}-c$

$$= \frac{8}{3}-\frac{1}{3} = \frac{7}{3}$$

OR $\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3}-\frac{1^3}{3} = \frac{8}{3}-\frac{1}{3} = 7/3$

Properties of definite Integrals:

① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If we can interchange the limits on any definite integral, then we need to do give a minus (−) sign onto the integral.

Ex:- $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2}-\frac{1^2}{2} = \frac{4}{2}-\frac{1}{2} = 2-\frac{1}{2} = \frac{3}{2}$

$\int_2^1 x\,dx = \left[\frac{x^2}{2}\right]_2^1 = \frac{1^2}{2}-\frac{2^2}{2} = \frac{1}{2}-\frac{4}{2} = \frac{1}{2}-2 = -3/2$

$\therefore \int_1^2 x\,dx = -\int_2^1 x\,dx$

② $\int_a^a f(x)\,dx = 0$ i.e. If the upper and lower limits are same then integral is zero.

Ex - $\int_2^2 x\,dx = \left[\frac{x^2}{2}\right]_2^2 = \frac{2^2}{2}-\frac{2^2}{2} = 0$

(3) $\int_a^b c f(x)dx = c\int_a^b f(x)dx$ where c is a constant. So take outside of integral when integrated.

Ex:- $\int_1^2 \underset{c}{2}x\,dx = \left[\frac{2x^2}{2}\right]_1^2 = [x^2]_1^2 = 2^2-1^2 = 3$

$2\int_1^2 x\,dx = 2\left[\frac{x^2}{2}\right]_1^2 = 2\left(\frac{2^2}{2}-\frac{1^2}{2}\right) = 2\times 3/2 = 3$

$\therefore \int_1^2 2x\,dx = 2\int_1^2 x\,dx$

(4) $\int_a^b (f(x)\pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

i.e. we can break up the definite integrals across a sum or difference

Ex:- $\int_1^2 (x+e^x)dx = \left[x^2+e^x\right]_1^2 = 2^2+e^2-1-e = 3+e^2-e$

$\int_1^2 x\,dx + \int_1^2 e^x dx = \left[\frac{x^2}{2}\right]_1^2 + [e^x]_1^2 = \frac{2^2}{2}-\frac{1^2}{2}+e^2-e$

$= 3+e^2-e$

$\therefore \int_1^2 (x+e^x)dx = \int_1^2 x\,dx + \int_1^2 e^x dx$

(5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where c is any constant and $a<c<b$

Ex: $\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{3^2}{2}-\frac{1^2}{2} = \frac{9}{2}-\frac{1}{2} = \frac{8}{2} = 4$

$1<2<3$

$\int_1^2 x\,dx + \int_2^3 x\,dx = \left[\frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 = \cancel{\frac{2^2}{2}}-\frac{1^2}{2}+\frac{3^2}{2}-\cancel{\frac{2^2}{2}}$

$= \frac{9}{2}-\frac{1}{2} = \frac{8}{2} = 4$

$\therefore \int_1^3 x\,dx = \int_1^2 x\,dx + \int_2^3 x\,dx$

(6) $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy$

i.e. Definite integrals are independent of variables with same limits (doesnot affect the answer)

$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2}-\frac{1^2}{2} = 3/2$ $\therefore$ ~~$\int_a^b f(x)dx$~~

$\int_1^2 t\,dt = \left[\frac{t^2}{2}\right]_1^2 = \frac{2^2}{2}-\frac{1^2}{2} = 3/2$

$$\therefore \int_1^2 x\,dx = \int_1^2 t\,dt$$

(7) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Ex $$\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - 0 = 2$$

$$\int_0^2 (2-x)\,dx = -\int_2^0 t\,dt = \int_0^2 t\,dt \qquad \left|\begin{array}{l} 2-x = t \\ \Rightarrow -dx = dt \end{array}\right.$$

$$= \left(\frac{t^2}{2}\right)_0^2 = \frac{2^2}{2} - 0 = 2$$

$$\therefore \int_0^2 x\,dx = \int_0^2 (2-x)\,dx$$

(8) $$\int_{-a}^a f(x)\,dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \quad [f(x) \text{ is odd fn}] \\ 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \quad [f(x) \text{ is even fn}] \end{cases}$$

Ex: $$\int_{-2}^2 x\,dx = \int_{-2}^0 x\,dx + \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_{-2}^0 + \left(\frac{x^2}{2}\right)_0^2$$

$$= 0 - \frac{(-2)^2}{2} + \frac{2^2}{2} - 0 = -\frac{4}{2} + \frac{4}{2} = 0$$

since $f(x) = x \Rightarrow f(-x) = -x = -f(x) \Rightarrow f(x)$ is odd

$$\Rightarrow \int_{-2}^2 x\,dx = 0$$

For even $$\int_{-2}^2 x^2\,dx = \int_{-2}^0 x^2\,dx + \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_{-2}^0 + \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{0^3}{3} - \frac{(-2)^3}{3} + \frac{2^3}{3} - \frac{0^3}{3} = -\left(-\frac{8}{3}\right) + \frac{8}{3}$$

$$= 2 \times \frac{8}{3} = \frac{16}{3}$$

$$\therefore 2\int_0^2 \cancel{f(x)}\,x^2\,dx = 2\int_0^2 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^2 = 2\left[\frac{2^3}{3} - \frac{0}{3}\right]$$

$$= 2 \times \frac{8}{3} = \frac{16}{3}$$

$$\therefore \int_{-2}^2 x^2\,dx = 2\int_0^2 x^2\,dx$$

Also here $f(x) = x^2 \Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is even.

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$\therefore \int_{-2}^{2} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) = x \text{ (odd)} \\ 2\int_0^2 f(x)\,dx & \text{if } f(x) = x^2 \text{ (even)} \end{cases}$

(9) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

Ex: $\int_0^{2\cdot 1} x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2$

$2\int_0^1 x\,dx = 2\cdot\left[\frac{x^2}{2}\right]_0^1 = 2\times\frac{1}{2} = 1$ if $f(2-x) = f(x)$

if $2-x = x$

OR : $\int_0^{\pi} \sin x\,dx = \int_0^{2\cdot\pi/2} \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx$

as $\sin(\pi-x) = \sin x$

$\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi} = 0$ as $\cos(\pi-x) = -\cos x$

Problems:

(1) $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$

$= \tan^{-1}(\tan(\pi/4)) - \tan^{-1}\tan 0$

$= \pi/4 - 0 = \pi/4$

(2) $\int_0^{\pi/2} x\cos x\,dx$

$\int_0^{\pi/2} x\cos x = x\int \cos x\,dx - \int\left(\frac{d}{dx}(x)\cdot\int\cos x\,dx\right)dx$

$= x\cdot\sin x - \int 1\cdot\sin x\,dx$

$= x\sin x + \cos x + c$

$\therefore \int_0^{\pi/2} x\cos x = \left[x\sin x + \cos x\right]_0^{\pi/2}$

$= \left[\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - [0\cdot\sin 0 + \cos 0]\right]$

$= \frac{\pi}{2}\cdot 1 + 0 - 0 - 1 = \frac{\pi}{2} - 1$

(3) $\int_{-2}^{2} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_0^2 |x|\,dx$

$= \int_{-2}^{0} -x\,dx + \int_0^2 x\,dx$

-ve 0 +ve, -2 2

as $|x| = \begin{cases} -x & x<0 \\ x, & x\geq 0 \end{cases}$

$$= \int_0^{-2} x\,dx + \int_0^{2} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^{-2} + \left[\frac{x^2}{2}\right]_0^{2} = \frac{(-2)^2}{2} - 0 + \frac{2^2}{2} - 0 = \frac{4}{2}+\frac{4}{2} = 4$$

(4) $$\int_2^5 [x]\,dx = \int_2^3 [x]\,dx + \int_3^4 [x]\,dx + \int_4^5 [x]\,dx$$

$$= \int_2^3 2\,dx + \int_3^4 3\,dx + \int_4^5 4\,dx$$

$$= 2[x]_2^3 + 3[x]_3^4 + 4[x]_4^5$$

$$= 2(3-2) + 3(4-3) + 4(5-4)$$

$$= 2+3+4 = 9$$

(5) $$\int_0^1 x \log(1+x)\,dx$$

$$\int \underset{2}{x}\,\underset{1}{\log(1+x)}\,dx = \log(1+x)\int x\,dx - \int\left(\frac{d}{dx}(\log(1+x))\int x\,dx\right)dx$$

$$= \log(1+x)\cdot\frac{x^2}{2} - \int \frac{1}{1+x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int \frac{x^2}{1+x}\,dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\left[x - \frac{x}{1+x}\right]dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\int x\,dx + \frac{1}{2}\int\frac{x}{1+x}\,dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\int\frac{1+x-1}{1+x}\,dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{x^2}{4} + \frac{1}{2}\int\left(1 - \frac{1}{1+x}\right)dx$$

$$= \frac{x^2}{2}\log(1+x) - \frac{x^2}{4} + \frac{1}{2}\int dx - \frac{1}{2}\int\frac{dx}{1+x}$$

$$I = \frac{x^2}{2}\log(1+x) - \frac{x^2}{4} + \frac{1}{2}x - \frac{1}{2}\log|x+1| + C$$

$$\therefore \int_0^1 x\log(1+x)\,dx = \left[\frac{x^2}{2}\log(1+x) - \frac{x^2}{4} + \frac{1}{2}x - \frac{1}{2}\log|x+1|\right]_0^1$$

$$= \frac{1^2}{2}\log(1+1) - \frac{1^2}{4} + \frac{1}{2}\cdot 1 - \frac{1}{2}\log(1+1) - \left[\frac{0^2}{2}\log(1+0) - \frac{0^2}{4} + \frac{1}{2}0 - \frac{1}{2}\log(0+1)\right]$$

$$= \frac{1}{2}\cancel{\log 2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2}\cancel{\log 2} + \frac{1}{2}\log 1$$

$$= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$\therefore \int_0^1 x\log(1+x)\,dx = \frac{1}{4}$$

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(6) Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx = I$

$\therefore I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x}$

$= \int_0^{\pi/2} \left(\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\sin x + \cos x}\right) dx \qquad \left(\because \int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right)$

$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$

$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2 - 0 = \pi/2$

$\Rightarrow I = \pi/2 \times \frac{1}{2} = \pi/4$

$\therefore \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \pi/4$

(7) Evaluate $\int_0^{\pi/2} \log \tan x\, dx$

Let $I = \int_0^{\pi/4} \log \tan x\, dx. \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \log \tan(\pi/2 - x)\, dx$

$= \int_0^{\pi/2} \log \cot x\, dx = I$

$\therefore I + I = \int_0^{\pi/2} \log \tan x\, dx + \int_0^{\pi/2} \log \cot x\, dx$

$= \int_0^{\pi/2} (\log \tan x + \log \cot x)\, dx$

$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx$

$\Rightarrow 2I = \int_0^{\pi/2} \log 1\, dx = 0$

$\Rightarrow I = 0$

$\therefore \int_0^{\pi/2} \log \tan x\, dx = 0$

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(8) Evaluate $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan x} = \int_0^{\pi/2} \frac{1}{1+\tan(\pi/2 - x)}\,dx$ $\left[\text{as } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \frac{1}{1+\cot x}\,dx = I$

$\therefore I + I = \int_0^{\pi/2} \frac{dx}{1+\tan x} + \int_0^{\pi/2} \frac{dx}{1+\cot x}$

$= \int_0^{\pi/2} \left(\frac{1}{1+\tan x} + \frac{1}{1+\cot x}\right) dx$

$= \int_0^{\pi/2} \left(\frac{1}{1+\tan x} + \frac{1}{1+\frac{1}{\tan x}}\right) dx$

$= \int_0^{\pi/2} \left(\frac{1}{1+\tan x} + \frac{\tan x}{1+\tan x}\right) dx$

$= \int_0^{\pi/2} \left(\frac{1+\tan x}{1+\tan x}\right) dx = \int_0^{\pi/2} dx$

$\Rightarrow 2I = [x]_0^{\pi/2} = \pi/2 - 0 = \pi/2$

$\Rightarrow I = \pi/2 \times \frac{1}{2} = \pi/4$

$\therefore \int_0^{\pi/2} \frac{dx}{1+\tan x} = \pi/4$

(9) Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

Let $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = I$

$\therefore I + I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$

$= \int_0^{\pi/2} \left(\frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx = \int_0^{\pi/2} dx$

$\Rightarrow 2I = (x)_0^{\pi/2} = \pi/2 - 0 = \pi/2$

$\Rightarrow I = \pi/4$

$\therefore \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \pi/4$

(10) Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx$

Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$

$= \int_0^{\pi/4} \log(1+\tan(\pi/4 - x))\,dx \quad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$= \int_0^{\pi/4} \log\left(1+\frac{\tan\pi/4 - \tan x}{1+\tan\pi/4 \cdot \tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{1+\cancel{\tan x}+1-\cancel{\tan x}}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \left(\log 2 - \log(1+\tan x)\right)dx$

$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$

$= \int_0^{\pi/4} \log 2\,dx - I$

$\Rightarrow I + I = \int_0^{\pi/4} \log 2\,dx = \log 2 \int_0^{\pi/4} dx$

$= \log 2\,[x]_0^{\pi/4} = \log 2\,(\pi/4 - 0)$

$\Rightarrow 2I = \pi/4 \log 2$

$\Rightarrow I = \frac{\pi}{8}\log 2$

$\therefore \int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$

(11)

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Application of Integration:

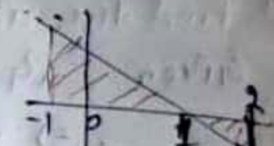
① Area :- Bounded by the curve:

(a) $\int_a^b f(x)\,dx$ = the area of the region bounded by the curve $y = f(x)$. w.r.t x-axis from a to b.

(b) $\int_a^b f(y)\,dy$ = the area of the region bounded by the curve $x = f(y)$ w.r.t y-axis from a to b.

Ex ① Find the area bounded by $x+y-1=0$ from -1 to 2

Given $x+y-1=0 \Rightarrow y = 1-x = f(x)$



$\therefore$ Area $= \int_{-1}^{1}(1-x)\,dx + (-)\int_{1}^{2}(1-x)\,dx$

$= \left[x - \frac{x^2}{2}\right]_{-1}^{1} - \left[x - \frac{x^2}{2}\right]_{1}^{2}$

$= \left(1-\frac{1}{2}\right) - \left(-1-\frac{1}{2}\right) = \left(2-\frac{4}{2}\right) - \left(1-\frac{1}{2}\right)$

$= \frac{1}{2} + \frac{3}{2} - \left(0 - \frac{1}{2}\right) = \frac{4}{2} + \frac{1}{2} = 2 + \frac{1}{2} = 5/2$ sq units (Ans)

② Find the area bounded by $y = e^x$ from 2 to 4.

Given $y = e^x$

$\therefore$ Area $= \int_2^4 e^x\,dx = [e^x]_2^4 = e^4 - e^2$ sq units

③ Find the area in the first quadrant bounded by $y = 4x^2$, $x=0$, $y=1$ and $y=4$.

Given $y = 4x^2 \Rightarrow x = \sqrt{\frac{1}{4}y} = \frac{1}{2}\sqrt{y}$

$x = 0 \rightarrow$ y-axis.

$\therefore$ Area $= \int_1^4 x\,dy = \int_1^4 \frac{1}{2}\sqrt{y}\,dy = \frac{1}{2}\int_1^4 \sqrt{y}\,dy$

$= \frac{1}{2}\left[\frac{y^{3/2}}{3/2}\right]_1^4 = \frac{1}{2} \times \frac{2}{3}\left[y^{3/2}\right]_1^4$

$= \frac{1}{3}\left(4^{3/2} - 4^{1/2}\right) = \frac{1}{3}(8-1)$

$= \frac{7}{3}$ sq units.

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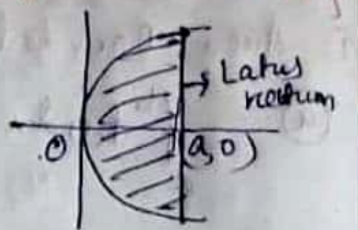
④ Find the area of the parabola $y^2=4ax$ bounded by its latus rectum.



Given $y^2=4ax \Rightarrow y=\sqrt{4ax}=2\sqrt{ax}$

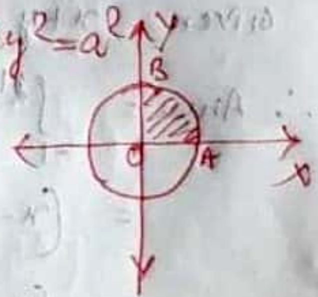
$\therefore$ Area $= 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{ax}\,dx$

$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$

$= 4\sqrt{a}\times\frac{2}{3}\cdot\left(a^{3/2}-a\right)$

$= \frac{8}{3}\sqrt{a}\cdot a^{3/2} = \frac{8}{3}a^2$ sq units.

⑤ Find the area of the circle $x^2+y^2=a^2$



Given $x^2+y^2=a^2$

$\Rightarrow y=\pm\sqrt{a^2-x^2}$

In the 1st quadrant, $y>0$,

we have $y=\sqrt{a^2-x^2}$

$\therefore$ Required area $= 4\times$ area of OAB

$= 4\int_0^a y\,dx = 4\int_0^a \sqrt{a^2-x^2}\,dx$

Putting $x=a\sin\theta$

$dx = a\cos\theta\,d\theta$

As $x=0 \Rightarrow \theta=0$

$x=a \Rightarrow \theta=\pi/2$

$= 4\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$

$= 4\int_0^{\pi/2}\sqrt{a^2(1-\sin^2\theta)}\;a\cos\theta\,d\theta$

$= 4\int_0^{\pi/2}\sqrt{a^2\cos^2\theta}\;a\cos\theta\,d\theta$

$= 4\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta$

$= 4\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = 4a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$

$= 4a^2\int_0^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)d\theta = \frac{4a^2}{2}\left[\int_0^{\pi/2}d\theta+\int_0^{\pi/2}\cos2\theta\,d\theta\right]$

$= 2a^2\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\pi/2}$

$= 2a^2\left[\pi/2-0+\frac{\sin2\pi/2}{2}-\frac{\sin0}{2}\right]$

$= 2a^2\left[\pi/2+0-0\right] = 2a^2\pi/2 = \pi a^2$ sq unit.

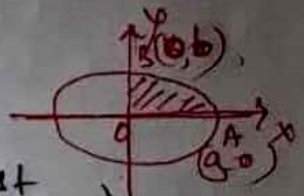
$\therefore$ Area $=\pi a^2$ sq units.

(6) Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = b^2\left(\frac{a^2-x^2}{a^2}\right)$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$ (+ve as in 1st quadrant)



$\therefore$ Required Area $= 4 \times$ Area AOB

$= 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

Substitute $x = a\sin\theta$
$\Rightarrow dx = a\cos\theta\,d\theta$
when $x=0 \Rightarrow \theta = 0$
$x = a \Rightarrow \theta = \pi/2$

$= \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$= \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta\,d\theta$

$= \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta$

$= \frac{4b}{a}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$

$= 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \frac{4ab}{2}\left[\int_0^{\pi/2} d\theta + \int_0^{\pi/2}\cos 2\theta\right]$

$= 2ab\left[[\theta]_0^{\pi/2} + \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\right]$

$= 2ab\left[\left(\pi/2 - 0\right) + \left\{\frac{\sin 2\pi/2}{2} - 0\right\}\right]$

$= 2ab \cdot \pi/2$

$= \pi ab$ sq units

$\therefore$ Area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq units.

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Differential Equation: (DE)

→ A differential equation is an equation with a function and one or more of its derivatives.

OR The equation of the form $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots \frac{d^ny}{dx^n}\right) = 0$

Ex:- $y + \frac{dy}{dx} = 5x$

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x$.

Two types: (Depending on derivation)

① Ordinary differential Equations:

An equation involving only one independent variable derivative and derivatives are called ordinary differential equations.

eg:- $3y + \frac{dy}{dx} = e^x$

② Partial differential Equation:

An equation with more than one independent variables and the derivatives occurring are partial, are called partial differential equations.

eg: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Order of the Differential Equation:

The order of a differential equation is the highest order of the derivatives present in the equation:

Ex. ① $\frac{dy}{dx} + x^2 = 1 \Rightarrow$ order $= 1$

② $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 5x \Rightarrow$ order $= 2$

③ $\frac{d^3y}{dx^3} = 0 \Rightarrow$ order $= 3$.

Degree of the Differential Equation:

The degree of a differential equation is the highest exponent (power) of the highest order derivative after the equation has been freed from radicals and fractions as far as derivatives are concerned.

① $\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x \Rightarrow$ degree $= 2$

② $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = 0 \Rightarrow$ degree $= 3$

(3) $\frac{d^2y}{dx^2} = \sqrt{3+\frac{dy}{dx}}$ (removing radical sign of RHS)

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(\sqrt{3+\frac{dy}{dx}}\right)^2$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = 3+\frac{dy}{dx}$ $\Rightarrow$ degree $= 2$

(4) $\left(1+\left(\frac{dy}{dx}\right)^2\right)^{5/2} = 3\left[\frac{d^2y}{dx^2}\right]$ (exponents to be non fractional)

$\Rightarrow \left(\left(1+\left(\frac{dy}{dx}\right)^2\right)^{5/2}\right)^2 = \left(3\frac{d^2y}{dx^2}\right)^2$

$\Rightarrow \left(1+\left(\frac{dy}{dx}\right)^2\right)^5 = 9\left(\frac{d^2y}{dx^2}\right)^2$ $\Rightarrow$ degree $= 2$

Linear and Non-linear Differential Equation:

A differential equation in which the dependent variable and all its derivatives occur in the first degree only and are not multiplied together, is called a linear differential equation. Otherwise it is called non-linear differential equation.

Ex:- (i) $\frac{dy}{dx} + 5y = e^x$

(ii) $\frac{d^2y}{dx^2} = \sin x$

(iii) $\frac{d^2y}{dx^2} + x\frac{dy}{dx}$

$\rightarrow$ linear differential Equations

(iv) $\left(\frac{dy}{dx}\right)^2 = x+y$

(v) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

$\rightarrow$ Non-linear differential equations.

Solution of a Differential Equation:

The solution of a general ordinary differential equation of the nth order $F\left(x, y, \frac{dy}{dx}, \ldots \frac{d^ny}{dx^n}\right) = 0$ is the relation between the dependent variables, not involving derivatives and having the number of arbitrary constants which is equal to the ~~numbs~~ the order of the differential equation and satisfying the given differential equation

Explicit Solution:

$y = f(x)$ is a explicit solution of the differential equation

i.e $F(x, f(x), f'(x) \dots f^{(n)}(x)) = 0$

Implicit Solution:

The solution of the form $f(x, y) = 0$ is called implicit satisfied solution of the differential equation.

General Solution:

The Solution which contains the number of arbitrary constants equal to the order of the differential equation

Ex: $\frac{dy}{dx} + y = x$

Since Order is 1. Solution $y = Ax$ (Suppose) $A =$ Constant

Particular Solution:

A solution is obtained from the general solution by giving particular values to the arbitrary constant.

Ex Putting $A = 1$ in above solution $y = x$ is a particular solution.

Types of differential Equation: Depending on the order and degree of DE.

(1) First Order and First Degree DE:

The DE: $\frac{dy}{dx} = f(x)$

$\Rightarrow dy = f(x)\, dx$

Integrating both sides

$\int dy = \int f(x)\, dx$

$\Rightarrow y = F(x) + C$ where $\int f(x)\, dx = F(x) + C$

is the general solution.

i.e The solution of a DE of 1st order and 1st degree contains only one arbitrary constant.

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①

② Second Order and First degree DE :

The equation is

$$\frac{d^2y}{dx^2} = f(x)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x) \qquad \left(\text{Let } p = \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dp}{dx} = f(x)$$

$$\Rightarrow dp = f(x)\,dx$$

Integrating both sides

$$\int dp = \int f(x)\,dx$$

$$\Rightarrow p = F(x) + C_1$$

$$\Rightarrow \frac{dy}{dx} = F(x) + C_1$$

$$\Rightarrow dy = (F(x) + C_1)\,dx$$

Again integrating both sides

$$\int dy = \int (F(x) + C_1)\,dx$$

$$\Rightarrow y = \int F(x)\,dx + \int C_1\,dx$$

$$\Rightarrow y = G(x) + C_1 x + C_2$$

is the general solution of DE having 2nd order and 1st degree with 2 arbitrary constants.

For Particular Solution:

Given $\frac{dy}{dx} = e^x$ when $x = 0 \Rightarrow y = 0$

$$\Rightarrow dy = e^x\,dx$$

Integrating both sides

$$\int dy = \int e^x\,dx$$

$$\Rightarrow y = e^x + c \quad \text{— ①}$$

putting $x = 0$ & $y = 0$ in eq ①

$$0 = e^0 + c \Rightarrow 0 = 1 + c \Rightarrow c = -1.$$

$\therefore$ Solution is $y = e^x - 1$ (Particular solution)

* The solution is obtained by removing the derivatives present in that equation.

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Formation of a Differential Equation:

1. Given the general solution of the DE.
2. Eliminating the arbitrary constants present in the equation by taking the derivatives.

(Order of DE = No. of arbitrary constants)

Ex:- Find the differential equation which has solution $y = e^{mx}$ where m is constant.

Given $y = e^{mx}$ ——— (1)

Differentiating wrt x on both sides of (1)

$$\frac{dy}{dx} = \frac{d}{dx}(e^{mx}) = me^{mx}$$

$$\Rightarrow \frac{dy}{dx} = my$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log y}{x} y$$

$$\Rightarrow x\frac{dy}{dx} = y\log y$$

is the required DE.

$\because y = e^{mx}$

$\Rightarrow \log y = mx$

$\Rightarrow \frac{\log y}{x} = m$

(2) Find the differential equation of $y = e^{x}(A\cos x + B\sin x)$

Given $y = e^{x}(A\cos x + B\sin x)$ ——— (1)

Differentiating wrt x in (1) we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x}(A\cos x + B\sin x)\right)$$

$$= e^{x}\frac{d}{dx}(A\cos x + B\sin x) + (A\cos x + B\sin x)\frac{d}{dx}(e^{x})$$

$$= e^{x}(-A\sin x + B\cos x) + (A\cos x + B\sin x)e^{x}$$

$$\Rightarrow \frac{dy}{dx} = e^{x}(-A\sin x + B\cos x) + y \quad \text{——— (2)}$$

$\Rightarrow$ Again differentiating wrt x in (2) both sides

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{x}(-A\sin x + B\cos x)\right\} + dy)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{x}(-A\sin x + B\cos x)\right) + \frac{dy}{dx}$$

$$= e^{x}\frac{d}{dx}(-A\sin x + B\cos x) + (-A\sin x + B\cos x)\frac{d}{dx}(e^{x}) + \frac{dy}{dx}$$

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(6)

$$\Rightarrow \frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + (-A\sin x + B\cos x)e^x + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = -e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = -y + \frac{dy}{dx} - y = -2y + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y - \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$ is the required DE.

(3) Find the differential equation of the family of circles with centre at the origin?

~~Given let the equation of circle be passing th~~

Let the equation of circle with centre at the origin be given by

$$x^2 + y^2 = r^2 \quad (\text{where } r = \text{radius} = \text{constant})$$

Differentiating wrt x on both sides

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(r^2)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} = 0$$ is the required DE.

(4) Find the differential equation of all straight lines passing through the point $(1,1)$.

The equation of straight line passing through the point $(1,1)$ and slope m is given by

$$y - 1 = m(x-1) \quad \text{where } m \text{ is constant}$$

Differentiating wrt x on both sides

$$\frac{d}{dx}(y-1) = \frac{d}{dx}(m(x-1))$$

$$\Rightarrow \frac{dy}{dx} = m.$$

$\therefore$ The required differential equation is $y - 1 = \frac{dy}{dx}(x-1)$

ie $y = (x-1)\frac{dy}{dx} + 1$

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Solution of the Differential Equation of First order and first degree:

Methods:-

① Separation of variables / Variable separable

Differential equations of the form
$$g(y)\,dy = f(x)\,dx$$
are called equations with separated variables.

→ The solutions of which are obtained by taking direct integration

i.e $\int g(y)\,dy = \int f(x)\,dx + C$

where C is a constant of integration.

Ex ① Solve $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Given $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ (Here, it already separated variables)

Integrating both sides
$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$
$$\Rightarrow \tan^{-1}y = \tan^{-1}x + C.$$

② Solve $\frac{dy}{dx} = (e^x+1)\,y$

Solⁿ: Given $\frac{dy}{dx} = (e^x+1)\,y$
$$\Rightarrow \frac{dy}{y} = (e^x+1)\,dx \quad \text{(Separated variables)}$$

Integrating on both sides
$$\int \frac{dy}{y} = \int (e^x+1)\,dx$$
$$\Rightarrow \log y = \int e^x\,dx + \int dx$$
$$\Rightarrow \log y = e^x + x + C.$$

③ Note, In 1st example, the solution is
$$\tan^{-1}y = \tan^{-1}x + C \qquad C = \tan^{-1}K$$
$$\Rightarrow \tan^{-1}y - \tan^{-1}x = \tan^{-1}K \qquad = \text{Constant}$$
$$\Rightarrow \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1}K \Rightarrow \frac{y-x}{1+xy} = K$$

(8)

(3) Solve $x(1+y^2)dx + y(1+x^2)dy = 0$

Soln: Given $x(1+y^2)dx + y(1+x^2)dy = 0$

$\Rightarrow x(1+y^2)dx = -y(1+x^2)dy$

$\Rightarrow y(1+x^2)dy = -x(1+y^2)dx$

$\Rightarrow \frac{y}{1+y^2}dy = \frac{-x}{1+x^2}dx$ (Separated variables)

Integrating on both sides

$\int \frac{y}{1+y^2}dy = \int \frac{-x}{1+x^2}dx$

$\Rightarrow \frac{1}{2}\int \frac{\cancel{y}}{u}\frac{du}{\cancel{y}} = -\frac{1}{2}\int \frac{x}{v}\cdot\frac{dv}{x}$

$\Rightarrow \frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\int \frac{dv}{v}$

$\Rightarrow \frac{1}{2}\log u = -\frac{1}{2}\log v + c$

$\Rightarrow \frac{1}{2}\log(1+y^2) = -\frac{1}{2}\log(1+x^2) + c$

$\Rightarrow \log(1+y^2) + \log(1+x^2) = 2c$

$\Rightarrow \log((1+y^2)(1+x^2)) = 2c = \log k$ (say) as constant

$\Rightarrow (1+y^2)(1+x^2) = k$ is the required solution.

Substitute

$1+y^2 = u$

$\Rightarrow 2y\,dy = du$

$\Rightarrow dy = \frac{du}{2y}$

and

$1+x^2 = v$

$\Rightarrow 2x\,dx = dv$

$\Rightarrow dx = \frac{dv}{2x}$

(4) Solve $\frac{dx}{dy} + \sqrt{\frac{1-x^2}{1-y^2}} = 0$

Soln $\frac{dx}{dy} + \sqrt{\frac{1-x^2}{1-y^2}} = 0 \Rightarrow \frac{dx}{dy} = -\sqrt{\frac{1-x^2}{1-y^2}}$

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = -\frac{dy}{\sqrt{1-y^2}}$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0.$

Integrating both sides

$\int \frac{dy}{\sqrt{1-y^2}} + \int \frac{dx}{\sqrt{1-x^2}} = \int 0\,dx$

$\Rightarrow \sin^{-1}y + \sin^{-1}x = C$ is the required Soln

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⑤ Solve $\frac{dy}{dx} = 1+x+y+xy$

Soln: Given $\frac{dy}{dx} = 1+x+y+xy$

$$\Rightarrow \frac{dy}{dx} = (1+x) + y(1+x)$$

$$\Rightarrow \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \frac{dy}{1+y} = (1+x)\,dx$$

Integrating on both sides

$$\int \frac{dy}{1+y} = \int (1+x)\,dx$$

$$\Rightarrow \log|1+y| = x + \frac{x^2}{2} + C \quad \text{(Ans)}$$

⑥ Equation Reducible to variable separable :

The DE of the form $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to variable separable by the substitution : $ax+by+c = z$.

Ex: Solve $\frac{dy}{dx} + 1 = e^{x+y}$

Soln Given $\frac{dy}{dx} + 1 = e^{x+y}$

Substitute

$$x+y = z$$

$$\Rightarrow \frac{d}{dx}(x+y) = \frac{dz}{dx}$$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\Rightarrow \frac{dz}{dx} \cancel{-1} \cancel{+1} = e^{z}$$

$$\Rightarrow \frac{dz}{dx} = e^{z}$$

$$\Rightarrow e^{-z}\,dz = dx$$

Integrating on both sides

$$\int e^{-z}\,dz = \int dx$$

$$\Rightarrow -e^{-z} = x + C$$

$$\Rightarrow -1 = e^{z}(x+C)$$

$$\Rightarrow -1 = e^{x+y}(x+C)$$

$$\Rightarrow e^{x+y}(x+C) + 1 = 0 \quad \text{(Ans)}$$

(2) Linear Differential Equation: (LDE)

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a linear differential equation of first order.

Ex: $\frac{dy}{dx} + 5y = x^3$ is a linear DE of order 1.

Procedure for solving $\frac{dy}{dx} + P(x).y = Q(x)$

(i) Find integrating factor (I.F) $= e^{\int P(x)dx}$

(ii) Then the solution is given by

$$y \times (I.F) = \int [Q(x) \times (I.F)] dx + C$$

OR If the LDE is $\frac{dx}{dy} + P(y)x = Q(y)$, Then

(i) Find integrating factor (I F) $= e^{\int P(y)dy}$

(ii) Then the solution is given by

$$x \times (IF) = \int [Q(y) \times (I F)] dy + C$$

Examples:

(1) Solve $\frac{dy}{dx} + \frac{1}{x}y = 3x$

Soln -> Given $\frac{dy}{dx} + \frac{1}{x}y = 3x$ is a LDE

we have $P(x) = \frac{1}{x}$, $Q(x) = 3x$

$\therefore IF = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

So, the solution is given by

$$y.x = \int 3x.x\, dx + C$$

$$= \int 3x^2 dx + C$$

$$= 3\int x^2 dx + C$$

$$\Rightarrow xy = \cancel{3}\frac{x^3}{\cancel{3}} + C$$

$$\Rightarrow xy = x^3 + C \quad (Ans)$$

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(2) Solve $\frac{dy}{dx} + (\sec x) y = \tan x$

Sol^n:- Given $\frac{dy}{dx} + (\sec x) y = \tan x$ which is a LDE with $P(x) = \sec x$, $Q(x) = \tan x$

$$\therefore I.F = e^{\int \sec x \, dx} = e^{\log(\sec x + \tan x)} = (\sec x + \tan x)$$

Then the solution is given by

$$y.(\sec x + \tan x) = \int \tan x (\sec x + \tan x) dx + C$$

$$= \int (\sec x . \tan x + \tan^2 x) dx + C$$

$$= \int \sec x . \tan x \, dx + \int (\sec^2 x - 1) dx + C$$

$$= \sec x + \int \sec^2 x \, dx - \int dx + C$$

$$\Rightarrow y (\sec x + \tan x) = \sec x + \tan x - x + C$$

(3) Solve $(x+1) \frac{dy}{dx} - y = e^x (x+1)^2$

Sol^n:- Given $(x+1) \frac{dy}{dx} - y = e^x (x+1)^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x+1} = e^x (x+1)$$ ($(x+1)$ is divided throughout)

which is a LDE with

$P(x) = -\frac{1}{x+1}$, $Q(x) = e^x (x+1)$

$$\therefore I.F = e^{\int -\frac{1}{x+1} dx} = e^{-\int \frac{dx}{1+x}} = e^{-\log(1+x)} = e^{\log(1+x)^{-1}}$$

$$= (1+x)^{-1} = \frac{1}{1+x}$$

So, the solution is given by

$$y . \frac{1}{1+x} = \int e^x (x+1)^{\cancel{2}} \frac{1}{\cancel{1+x}} dx + C$$

$$= \int e^x (x+1) dx + C$$

$$= \int x e^x dx + \int e^x dx + C$$

$$= x e^x - \cancel{e^x} + \cancel{e^x} + C$$

(dx)

$$y/_{1+x} = x e^x + c$$

(4) Solve $x\log x \frac{dy}{dx} + y = \log x^2$

Soln: Given $x\log x \cdot \frac{dy}{dx} + y = \log x^2$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{\log x^2}{x\log x} = \frac{2\log x}{x\log x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x\log x}\right) y = \frac{2}{x}$$

which is a LDE with

$P(x) = \frac{1}{x\log x}$ and $Q(x) = \frac{2}{x}$

$$\therefore I.F = e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{dt}{t}}$$

$$= e^{\log t} = t = \log x$$

Substitute $\log x = t$

$\Rightarrow \frac{1}{x}dx = dt$

The solution is given by

$$y \cdot \log x = \int \frac{2}{x}\log x \cdot dx + C$$

$$= \int 2t\, dt + C$$

$$= 2\frac{t^2}{2} + C$$

$$\Rightarrow y\log x = (\log x)^2 + C$$

$$\Rightarrow y\log x = (\log x)^2 + C \quad \text{(Ans)}$$

(5) Solve $\frac{dx}{dy} + x\tan y = \sec y$

Soln: Given $\frac{dx}{dy} + x\tan y = \sec y$ which is a LDE

with $P(y) = \tan y$, $Q(y) = \sec y$

$$\therefore I.F = e^{\int \tan y\, dy} = e^{\log \sec y} = \sec y$$

The solution is given by

$$x \cdot \sec y = \int \sec y \cdot \sec y\, dy + C$$

$$= \int \sec^2 y\, dy + C$$

$$= \tan y + C$$

$$\Rightarrow x\sec y = \tan y + C \quad \text{(Ans)}$$

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Equations Reducible to the Linear form:

(a) DE of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

Can be reduced to the LDE form by dividing by y^n throughout and substituting $\frac{1}{y^{n-1}} = z$.

(b) DE of the form $f'(y)\frac{dy}{dx} + P(x) f(y) = Q(x)$

can be reduced to the LDE form by substituting $f(y) = z$.

Ex:- (1) Solve $\frac{dy}{dx} + xy = xy^3$

Soln:- Given $\frac{dy}{dx} + xy = xy^3$ which is no Linear.

Dividing by y^3 throughout

$$\frac{1}{y^3}\frac{dy}{dx} + x\frac{y}{y^3} = \frac{xy^3}{y^3}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + x\frac{1}{y^2} = x$$

$$\Rightarrow \frac{1}{\cancel{y^3}}\left(\frac{\cancel{y^3}}{2}\right)\cdot\frac{dz}{dx} + xz = x$$

$$\Rightarrow -\frac{1}{2}\frac{dz}{dx} + xz = x$$

$$\Rightarrow \frac{dz}{dx} - 2xz = -2x$$

Substitute $\frac{1}{y^2} = z$

$$\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$$

which is a Linear DE with

$P(x) = -2x$, $Q(x) = -2x$

$$\therefore I.F = e^{\int -2x\,dx} = e^{-2\int x\,dx} = e^{-\cancel{2}\times\frac{x^2}{\cancel{2}}} = e^{-x^2}$$

The solution is given by

$$z\cdot e^{-x^2} = \int(-2x)\cdot e^{-x^2}\,dx + c$$

$$= \int \cancel{(-2x)}\, e^t \frac{dt}{\cancel{-2x}} + c$$

$$= \int e^t\,dt + c$$

Substitute $-x^2 = t$

$\Rightarrow -2x\,dx = dt$

$\Rightarrow dx = \frac{dt}{-2x}$

$$\Rightarrow z e^{-x^2} = e^t + c$$

$$\Rightarrow \frac{1}{y^2}e^{-x^2} = e^{-x^2} + c \Rightarrow cy^2e^{x^2} + y^2 = 1 \text{ (ans)}$$

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(2) Solve $x\frac{dy}{dx} + y\log y = xye^x$

Soln :- Given $x\frac{dy}{dx} + y\log y = xye^x$ which is not a linear

Dividing by xy throughout we have

$$\frac{x}{xy}\frac{dy}{dx} + \frac{y\log y}{xy} = \frac{xye^x}{xy}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^x$$

Substitute $\log y = z$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \frac{dz}{dx} + \frac{1}{x}z = e^x$$

which is a LDE with

$P(x) = \frac{1}{x}$, $Q(x) = e^x$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

The solution is given by

$$z \cdot x = \int e^x \cdot x\, dx + C$$

$$= x\int e^x dx - \int\left(\frac{d}{dx}(x)\cdot\int e^x dx\right)dx + C$$

$$= xe^x - \int e^x dx + C$$

$$\Rightarrow zx = xe^x - e^x + C$$

$$\Rightarrow \log y\, x = xe^x - e^x + C$$

$$\Rightarrow x\log y = xe^x - e^x + C$$ (Ans)

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