LECTURE NOTES
ON
CONTROL SYSTEM ENGINEERING
6TH SEMESTER

PREPARED BY
MR. Nihar Ranjan Dikshit
PTGF IN ELECTRICAL ENGINEERING
GOVERNMENT POLYTECHNIC PURI

Signal Flow Graph

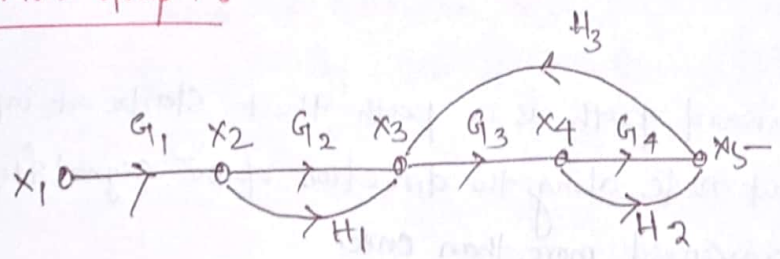
Introduction :

For a System with complex interrelations, the block diagram reduction procedure is complicated and often quite difficult to complete. An alternative method for determining the relationships between system variables has been developed by S. J. Mason.

Block diagrams and signal-flow graphs are closely related and accomplish the same purpose. But the signal flow graph has the advantage of more uniform notation. The signal flow graph is more convenient to draw and easier to visualize.

Signal flow graphs are an alternative to block-diagrams. Unlike block diagrams which consists of blocks, signals, summing points, and takeoff points, a signal flow graph consists only of branches, which represents systems and nodes, which represents signals.

Signal Flow Graph :



Node : A node of a signal flow graph is a point representing a system variable. It is denoted by a small dot (•) or a small circle (o). In fig X_1, X_2, X_3, X_4, X_5, are nodes.

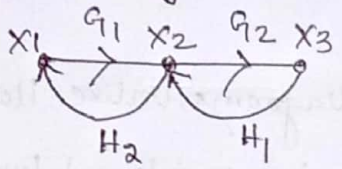
Branch : A Branch is a connection between two nodes of a network and represents the dependence of one node on the other.

Scanned by CamScanner

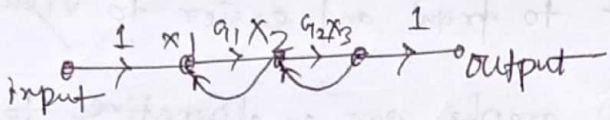
Classification of node :

There are three types of nodes.

1. Input node (Source node) – A node with only outgoing branches.
2. Output node (Sink node) – A node with only incoming branches.
3. Chain node (Mixed node) – A node with both incoming and outgoing branches.
4. Dummy node – If the incoming and outgoing branches exist at the first and the last node representing input and output variables, then separate input and output nodes are created by adding branches with gain '1'.



System without Input and output node.



System with dummy nodes.

Path : A path is a branch or a continuous unidirectional succession of branches, along which no node is traversed more than once.

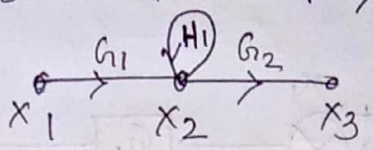
Forward path : A forward path is a path that starts at input node and ends at output node along the direction of the signal flow. no node should be traversed more than once.

Forward path Gain : The product of gains found by traversing a path from input to the output node of the signal flow graph in the direction of signal flow.

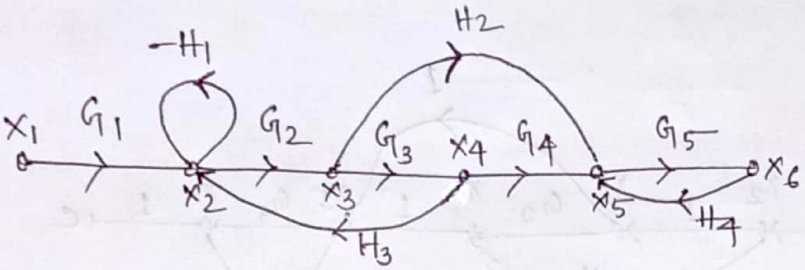
Loop : A closed path in the direction of arrows which originates and terminates at the same node and along which no node is traversed more than once.

Scanned by CamScanner

Self loop: A self loop is defined as a feedback path of only one branch which starts and terminates at the same node. example → H_1



Non touching Loops: Non-touching loops are such loops having no path or branches in common.



Loop-1 – $X_2 X_3 X_4 X_2$ Loop-2 – $X_5 X_6 X_5$ are non touching

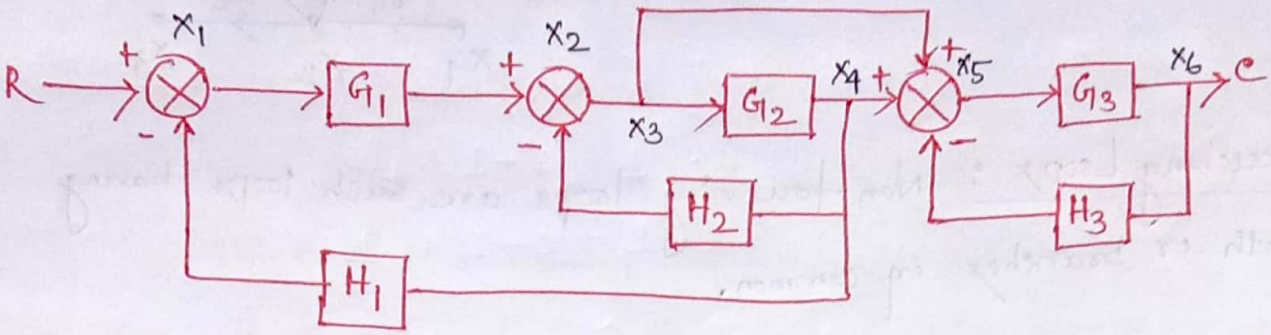
Similarly Loop-3 – $X_2 X_2$ Loop 2 – $X_5 X_6 X_5$ are non touching

Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node, following the direction of the signal flow without passing through any other node more than once.

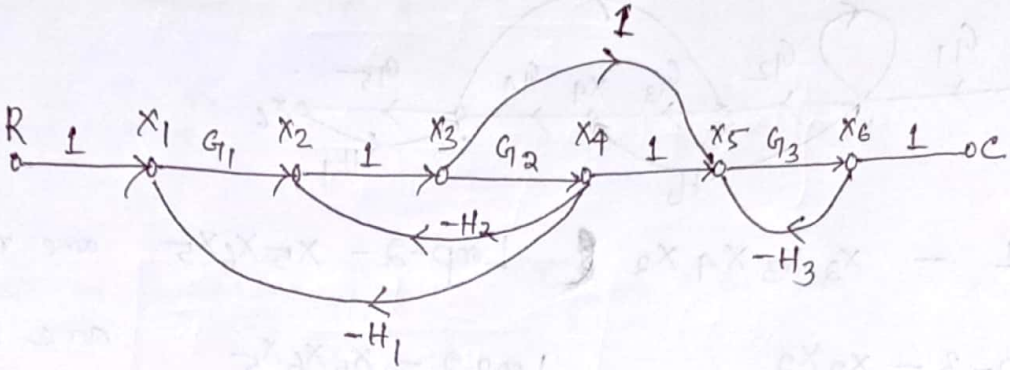
Gain of Loop-1 → $G_2 G_3 H_3$. Gain of Loop-2 → $G_5 H_4$ Gain of Loop-3 → $-H_1$

Scanned by CamScanner

Qus: Convert the block-diagram to signal flow graph.



Ans :—



Mason's Gain Formula :

The relationship between an input variable and an output variable of a signal flowgraph is given by the net gain between the input and output nodes. This gain is called the over all gain of the system.

We can find this over all gain by Mason's gain Formula

$$TF = \frac{C}{R} = \sum_{K=1}^{N} \frac{P_K \Delta_K}{\Delta}$$

Where N = no. of Forward paths

$K = 1, 2, 3 ---- N.$

P_K = Gain product of Kth forward path.

Δ = Determinant of signal flow graph.

Scanned by CamScanner

$\Delta = 1 - (\Sigma$ all loop gains$)$

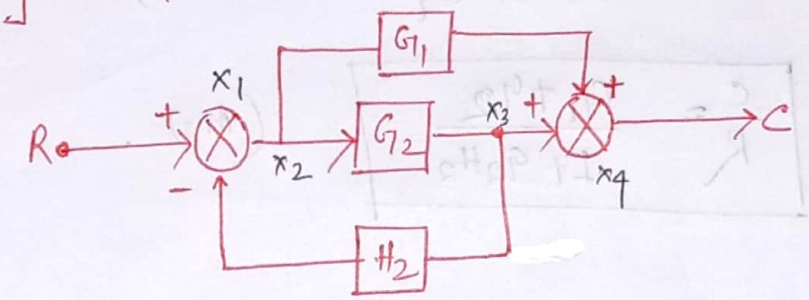
$+ (\Sigma$ products of gains of all possible of two nontouching loops$)$

$- (\Sigma$ products of " " " " " three " loops$)$

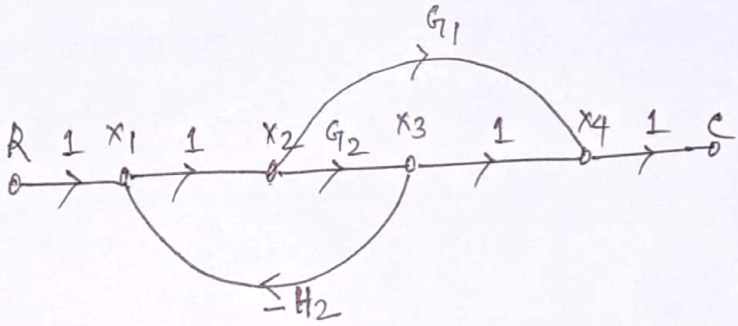
$+ \text{------}$

Δ_K = It is the value of Δ but excluding Kth forward path gain.

Qus:- Draw the signal flow graph and find the overall transfer function by Mason's Gain Formula.



Ans:- Step-1



Step -2 There are two forward paths

$P_1 = R \to x_1 \to x_2 \to x_3 \to x_4 \to C = 1 \times 1 \times G_2 \times 1 \times 1 = G_2$.

$P_2 = R \to x_1 \to x_2 \to x_4 \to C = 1 \times 1 \times G_1 \times 1 = G_1$

Step -3 There is one loop here

$L_1 = x_1 \to x_2 \to x_3 \to x_1 = 1 \times G_2 \times (-H_2) = -G_2 H_2$.

Scanned by CamScanner

Step-4 determinant of the SFG

$$\Delta = 1 - L_1 = 1 - (-G_2H_2) = 1 + G_2H_2 .$$

Step-5 Both the forward path touches the loop

$$\Delta_1 = 1 - 0 = 1 .$$

$$\Delta_2 = 1 - 0 = 1 .$$

Step-6 By Mason's Gain Formula

$$\frac{C}{R} = \frac{P_1\Delta_1}{\Delta} + \frac{P_2\Delta_2}{\Delta}$$

$$= \frac{1}{1+G_2H_2}\left\{G_1 \times 1 + G_2 \times 1\right\}$$

$$\boxed{\frac{C}{R} = \frac{G_1+G_2}{1+G_2H_2}}$$ (Ans).

Scanned by CamScanner

TIME RESPONSE ANALYSIS

Time response of a system is the output of the system as a function of time, when subjected to a known input.

Mainly time response is devided in to two types

- Transient response.
- Steady-state response.

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

where $C(t)$ = total time response.

$C_{tr}(t)$ = transient time response.

$C_{ss}(t)$ = steady-state response.

Standard test input :—

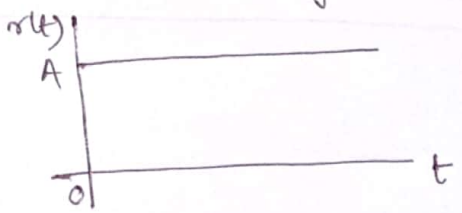
for design and analysis of a control system, some test signals are used as inputs. The most commonly used inputs for this purpose are

- Step function
- Ramp function
- parabolic function
- Impulse function.

Step function :—

Mathematically, the step function is given by

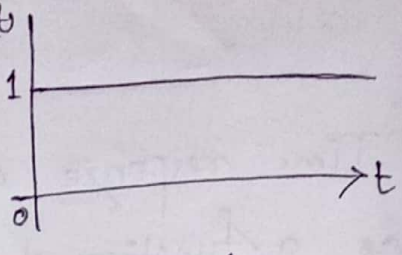
$$\left.\begin{aligned} r(t) &= 0 \text{ for } t < 0 \\ r(t) &= A \text{ for } t \geq 0 \end{aligned}\right\}$$



Scanned by CamScanner

if $A = 1$, the function $r(t) = u(t) = 1$.

it is known as unit step function.



The Laplace transform of unit step function is given by

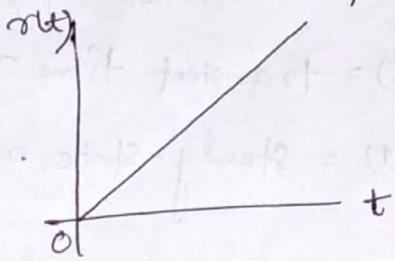
$$R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1\} = 1/s.$$

$$\boxed{R(s) = 1/s}$$

Ramp function:—

The ramp function can be mathematically represented by

$$\left.\begin{aligned} r(t) &= 0 \quad \text{for } t \le 0 \\ r(t) &= kt \quad \text{for } t > 0 \end{aligned}\right\}$$



where k = slope of the line

if $k = 1$, $r(t) = 1$.

Hence it is known as unit ramp function.

Laplace transform of unit ramp function

$$R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

$$\boxed{R(s) = \frac{1}{s^2}}$$

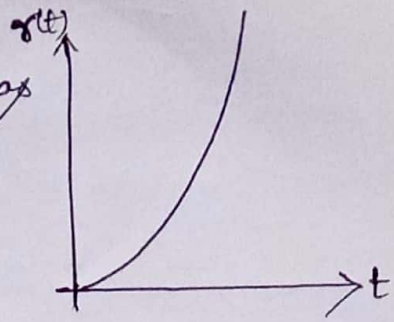
Ramp function is also called velocity function. It is noted that

$$\boxed{\int(\text{unit step function}) = \text{unit ramp function}}$$

Scanned by CamScanner

Parabolic function –

Mathematically this function can be represented as



$$\left.\begin{aligned} r(t) &= 0 \quad \text{for } t \le 0 \\ r(t) &= \tfrac{1}{2} t^2 \quad \text{for } t > 0 \end{aligned}\right\}$$

Laplace transform of unit parabolic function is

$$R(s) = \mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{s^3}$$

$$\boxed{R(s) = \frac{1}{s^3}}$$

∴ Parabolic function is also called acceleration function.

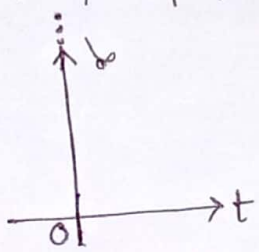
$$\boxed{\int \text{unit ramp function} = \text{unit parabolic function}}$$

Impulse Function

for an impulse function the duration (width) of the pulse approaches zero, the amplitude of the pulse approaches infinity, but the area of the pulse still remains unity.

Mathematically, a unit impulse function is defined as

$$\left.\begin{aligned} \delta(t) &= 0 \quad \text{for } t \ne 0 \\ \delta(t) &= \infty \quad \text{for } t = 0 \end{aligned}\right\}$$



Since area under the impulse is unity

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\, dt = 1}$$

$$\boxed{\int \text{impulse function} = \text{unit Step function.}}$$

∴ $\int \rightarrow$ Integration

∴ D → Derivative.

$$\text{Impulse} \xrightarrow{\int} \text{unit step} \xrightarrow{\int} \text{unit Ramp} \xrightarrow{\int} \text{Parabolic}$$

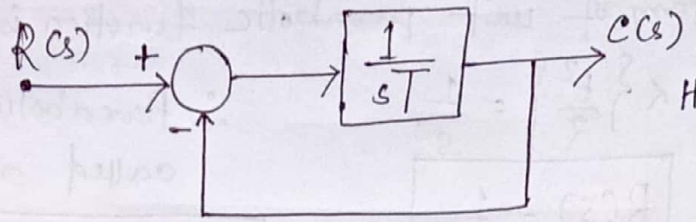
$$\text{Parabolic} \xrightarrow{D} \text{unit ramp} \xrightarrow{D} \text{unit Step} \xrightarrow{D} \text{Impulse}$$

Scanned by CamScanner

Time-Response for a first Order System.

First Order Control System:- A control system in which the highest power of 's' in the denominator of its transfer function is equal to '1'.

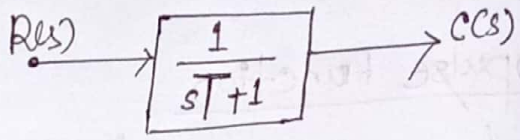
example:-



Here $G(s) = \frac{1}{sT}$

$H(s) = 1$

transfer function of the system



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1/sT}{1+(1/sT)(1)} = \frac{1}{sT+1}$$

Response of the first order system with unit step input

For a first order system $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

$$C(s) = R(s) \times \frac{1}{sT+1} \quad \text{——(eqn-1)}$$

for an unit step input $r(t) = 1$ and $R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1\} = \frac{1}{s}$

putting the value of $R(s)$ in (eqn-1)

$$C(s) = \frac{1}{s} \cdot \frac{1}{sT+1}$$

Expanding $C(s)$ in to partial fractions gives,

$$C(s) = \frac{1}{s} - \frac{T}{1+sT}$$

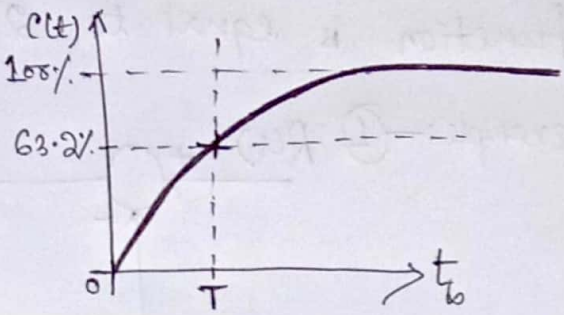
Scanned by CamScanner

$$C(s) = \frac{1}{s} - \frac{1}{s+(1/T)} \quad \text{——(eqn-2)}$$

Now taking inverse Laplace transform of eqn-2,

$$\mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+(1/T)}\right\}$$

$$c(t) = 1 - e^{-t/T}$$



At $t = T$, $c(t) = 1 - e^{-1} = 1 - 0.368 = 0.632$.

Here T is time Constant. It is defined as the time required for the output response to attain 63.2% of its final or steady state value.

Response of the first order System with unit impulse function

for unit impulse input

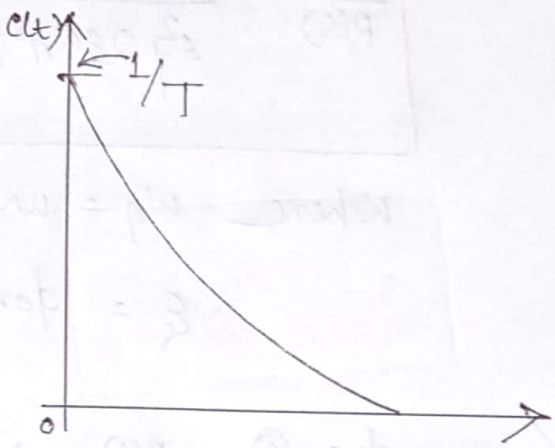
$$R(s) = 1.$$

$$C(s) = R(s) \times \frac{1}{sT+1} = 1 \cdot \frac{1}{sT+1}$$

$$C(s) = \frac{1}{s+\frac{1}{T}} \times \frac{1}{T}$$

$$c(t) = \frac{1}{T} \times e^{-t/T}.$$

$$c(0) = \frac{1}{T} \times e^{0} = \frac{1}{T} \times 1 = \frac{1}{T}$$

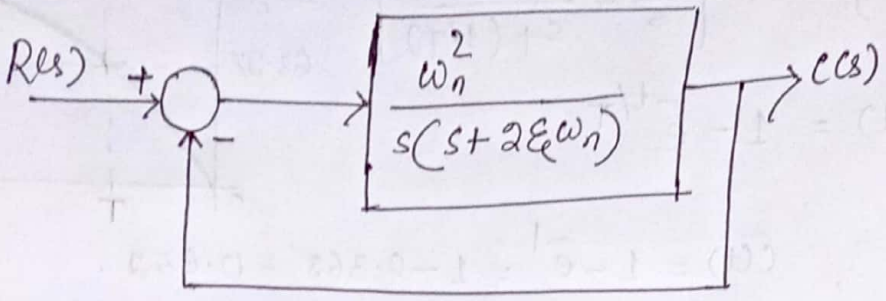


Scanned by CamScanner

Time Response for a Second order System.

Second Order System :- A control system in which the highest power of 's' in the denominator of its transfer function is equal to '2'.

example :- (1)



Here $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$, $H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2 / s(s+2\xi\omega_n)}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

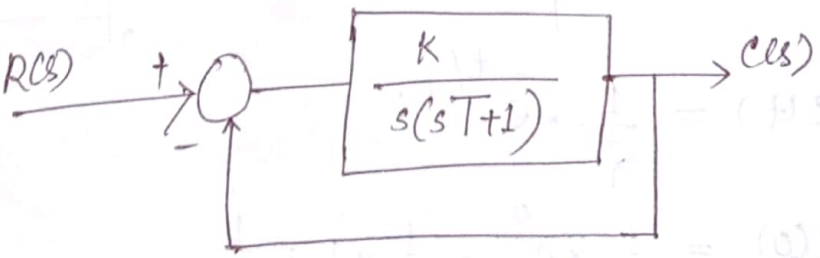
$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}$$

→ standard form of 2nd order closed loop control system.

Where ω_n = undamped natural frequency.

ξ = damping ratio (or) damping factor.

example :- (2)



Here $G(s) = \dfrac{K}{s(sT+1)}$, $H(s) = 1$

the transfer function is $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

Scanned by CamScanner

$$\boxed{\frac{C(s)}{R(s)} = \frac{K/T}{s^2 + \frac{1}{T}s + \frac{K}{T}}}$$

Comparing both transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{K/T}{s^2 + \frac{1}{T}s + \frac{K}{T}}$$

from the above equalities we can find

$\omega_n = \sqrt{\frac{K}{T}}$ → undamped natural frequency

$\xi = \frac{1}{2\omega_n T}$ → damping ratio.

$\frac{1}{\xi\omega_n}$ ⇒ 'time constant' of the system.

In the standard form of second order system, the denominator polynomial $s^2 + 2\xi\omega_n s + \omega_n^2$ is called 'characteristic polynomial' and $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ is called 'characteristic equation'.

as we can see the characteristic eqn is of a quadratic eqn. solving this eqn. we can get the roots of this eqn. as

$$\left.\begin{aligned} s_1 &= -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \\ s_2 &= -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \end{aligned}\right\} \text{poles of the transfer function}$$

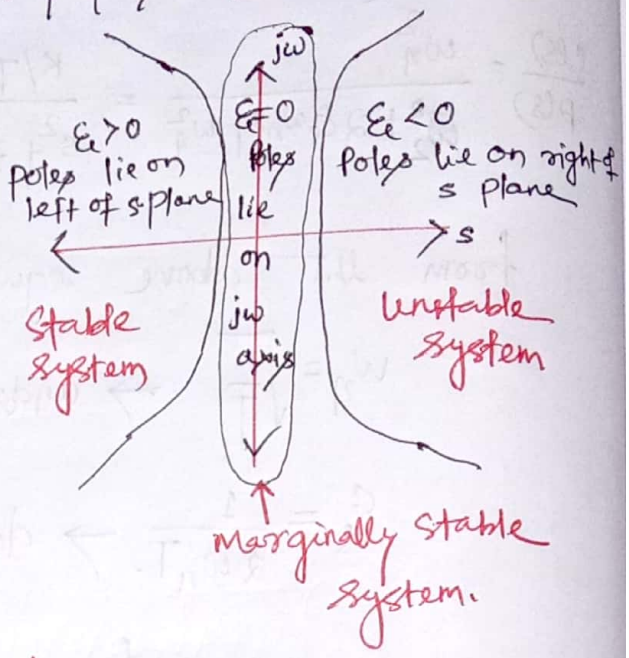
Scanned by CamScanner

now we have four different cases

i) $\xi = 0$ (undamped) → In this case transient response doesn't die out. system will be marginally stable.

ii) $0 < \xi < 1$ (underdamped) → closed loop poles s_1 and s_2 will be complex conjugates.

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$



Let $\omega_d = \omega_n\sqrt{1-\xi^2}$, Hence

$$s_{1,2} = -\xi\omega_n \pm j\omega_d.$$

iii) $\xi = 1$ (critically damped)

iv) $\xi > 1$ (over damped)

for $\xi < 0$ the system will be unstable.

1) Undamped Case ($\xi = 0$)

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s).$$

for a unit step function as an input

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

for undamped case $\xi = 0$, hence

$$C(s) = \frac{\omega_n^2}{(s^2 + \omega_n^2)s}$$

(for partial fraction use the form $\frac{A}{s} + \frac{Bs+C}{(s^2+\omega_n^2)}$)

Making partial fraction of $C(s)$ we can get

$$C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

Scanned by CamScanner

Taking the inverse laplace transform, we can get

$c(t) = 1 - \cos w_n t$ $\quad \therefore \mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$

to draw the graph for $c(t) = 1 - \cos w_n t$

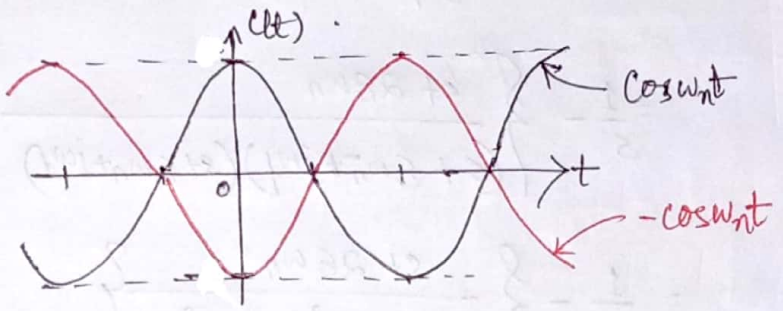
we have to follow three steps.

<u>Step- 1</u> draw for $\cos w_n t$.

<u>Step- 2</u> draw for $-\cos w_n t$ (reflection)

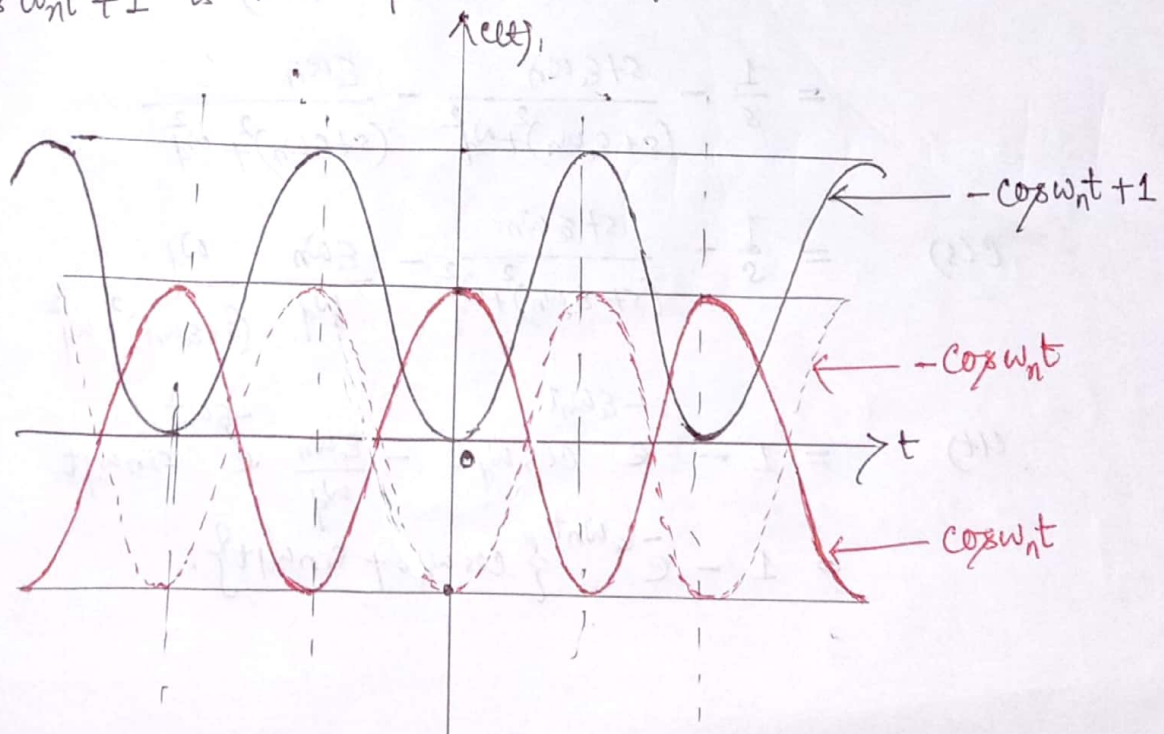
<u>Step- 3</u> draw for $-\cos w_n t + 1$. (vertical shift)

<u>Step-1</u> $\cos w_n t$



<u>Step-2</u> $-\cos w_n t$ is the reflection of $\cos w_n t$

<u>Step-3</u> $-\cos w_n t + 1$ is the upward shift of $-\cos w_n t$ function to one unit



Vertical shifting of $-\cos w_n t$ by adding 1.

Scanned by CamScanner

2) underdamped case ($0 < \xi < 1$)

Output of the system $C(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \times R(s)$

for a unit step function $r(t) = 1$ and $R(s) = \frac{1}{s}$.

Therefore, $C(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \times \dfrac{1}{s}$

After partial fraction we can get,

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \left\{ \frac{s + 2\xi\omega_n}{(s + \xi\omega_n - j\omega_d)(s + \xi\omega_n + j\omega_d)} \right\}$$

$$= \frac{1}{s} - \left\{ \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 - (j\omega_d)^2} \right\}$$

$$= \frac{1}{s} - \left\{ \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\}$$

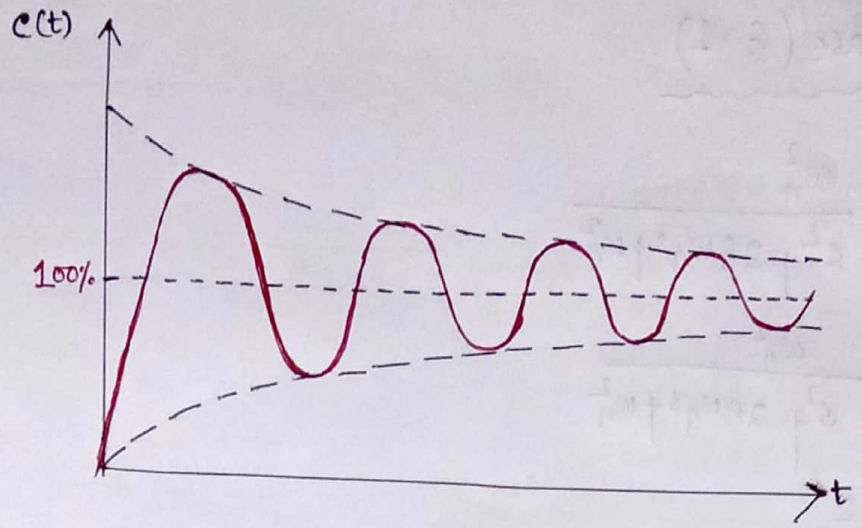
$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{\omega_d} \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\xi\omega_n t} \cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t} \sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t} \{ \cos\omega_d t + \sin\omega_d t \}.$$

Scanned by CamScanner



3) Critically damped case ($\xi = 1$)

If the two poles of $\frac{C(s)}{R(s)}$ are equal, the system is said to be critically damped one.

Putting $\xi = 1$ in $C(s)$ eqn. we can get

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\omega_n s + \omega_n^2)} R(s).$$

for unit step function as input $R(s) = 1/s$, hence

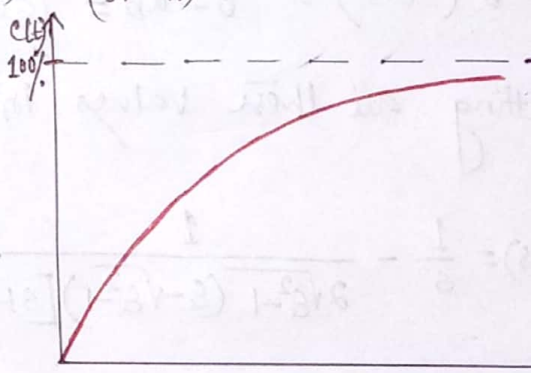
$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \times \frac{1}{s}$$

$$= \frac{\omega_n^2}{s(s+\omega_n)^2}$$

Partial fraction of $C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$

$$C(s) = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$\mathcal{L}^{-1}(C(s)) = c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$



Scanned by CamScanner

4) Overdamped Case ($\xi > 1$)

$$C(s) = R(s) \times \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} \times \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{s\left[s + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]\left[s + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]}$$

Let $\xi\omega_n + \omega_n\sqrt{\xi^2-1} = a$

$\xi\omega_n - \omega_n\sqrt{\xi^2-1} = b.$

$$C(s) = \omega_n^2 \left\{ \frac{1}{s} \times \frac{1}{(s+a)(s+b)} \right\}$$

After partial fraction we can get

$$C(s) = \omega_n^2 \left\{ \frac{1}{s(ab)} + \frac{1}{(-a)(s+a)(-a+b)} + \frac{1}{(-b)(s+b)(-b+a)} \right\}$$

now $ab = (\xi\omega_n)^2 - \left(\omega_n\sqrt{\xi^2-1}\right)^2 = \omega_n^2.$

$$-a(-a+b) = a^2 - ab = \left(\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)^2 - \omega_n^2 = 2\omega_n^2\sqrt{\xi^2-1}\left(\xi + \sqrt{\xi^2-1}\right)$$

$$-b(-b+a) = b^2 - ab = \left(\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)^2 - \omega_n^2 = 2\omega_n^2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)$$

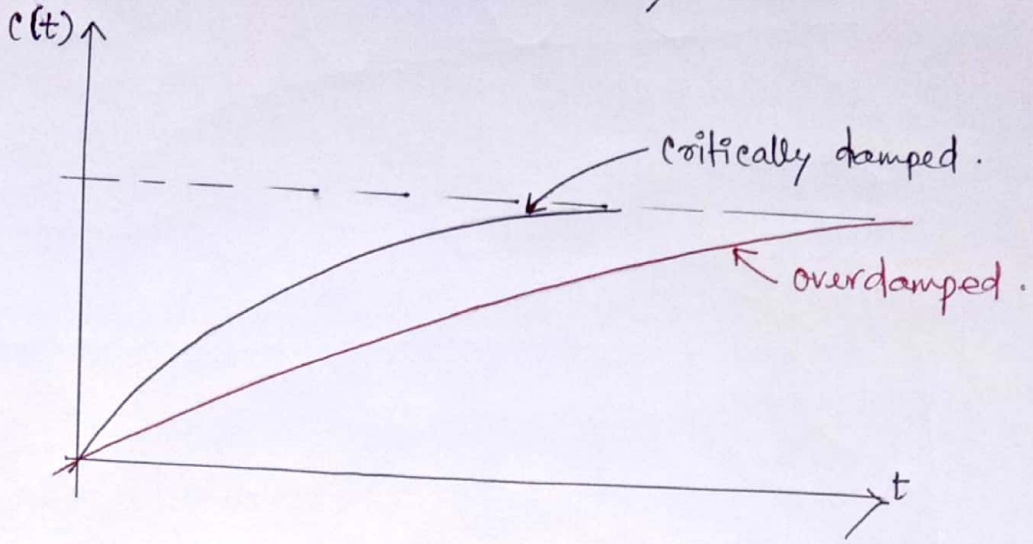
Putting all these values in $C(s)$ eqn.

$$C(s) = \frac{1}{s} - \frac{1}{2\sqrt{\xi^2-1}\left(\xi - \sqrt{\xi^2-1}\right)\left[s + \left(\xi - \sqrt{\xi^2-1}\right)\omega_n\right]} + \frac{1}{2\sqrt{\xi^2-1}\left(\xi + \sqrt{\xi^2-1}\right)\left[s + \left(\xi + \sqrt{\xi^2-1}\right)\omega_n\right]}$$

Scanned by CamScanner

$$\mathcal{L}^{-1}\{C(s)\} = c(t) = 1 + \frac{1}{2\sqrt{\xi^2-1}\,(\xi+\sqrt{\xi^2-1})}\, e^{-(\xi+\sqrt{\xi^2-1})t}$$

$$- \frac{1}{2\sqrt{\xi^2-1}\,(\xi-\sqrt{\xi^2-1})}\, e^{-(\xi+\sqrt{\xi^2-1})t}.$$



Scanned by CamScanner

Effect of ξ on pole location in second-order systems.

Case.1 $0 < \xi < 1$

a) complex conjugate poles.

$$s_1 = -\xi w_n + j w_n \sqrt{1-\xi^2}$$

$$s_2 = -\xi w_n - j w_n \sqrt{1-\xi^2}.$$

b) Poles are located on the second and third quadrant.

c) An underdamped response.

Case.2 $\xi = 1$

a) Poles are real and equal. $s_1 = s_2 = -\xi w_n$.

b) Poles lie on the -ve σ axis.

c) A critically damped response.

Scanned by CamScanner

Case-3 $\xi > 1$

a) poles are -ve, real, unequal.

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

b) poles lie on the -ve σ axis at unequal place

c) An overdamped response.

Case-4 $\xi = 0$

a) the poles are complex with only imaginary parts. They lie on the $j\omega$ axis. The poles are conjugates of each other.

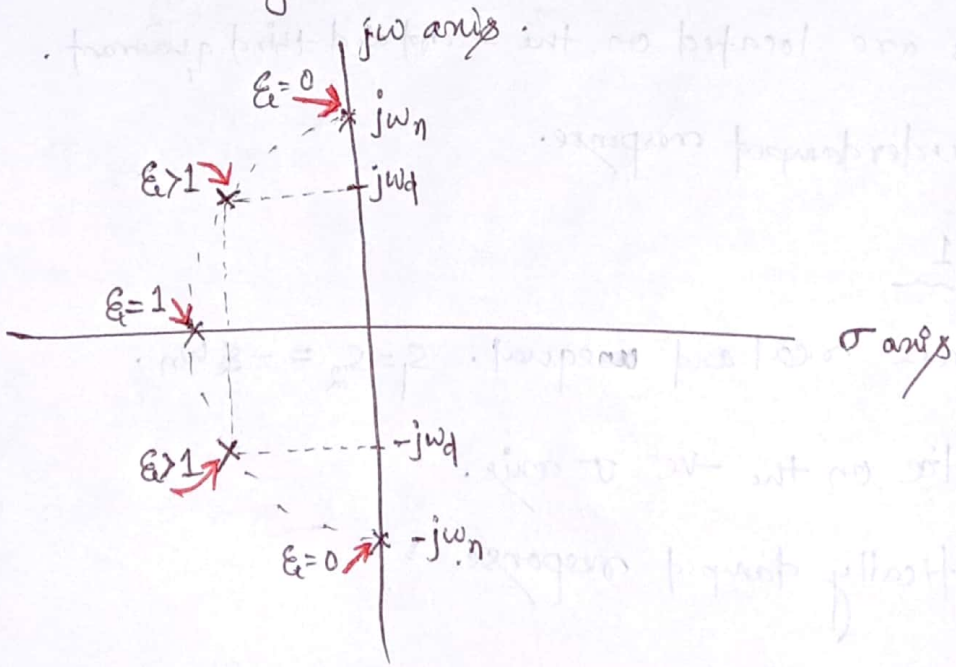
b) poles are given by $s_1 = -j\omega_n$
$s_2 = j\omega_n$

c) An undamped response.

Case-5 $\xi < 0$

a) the poles lie on the right half of s-plane.

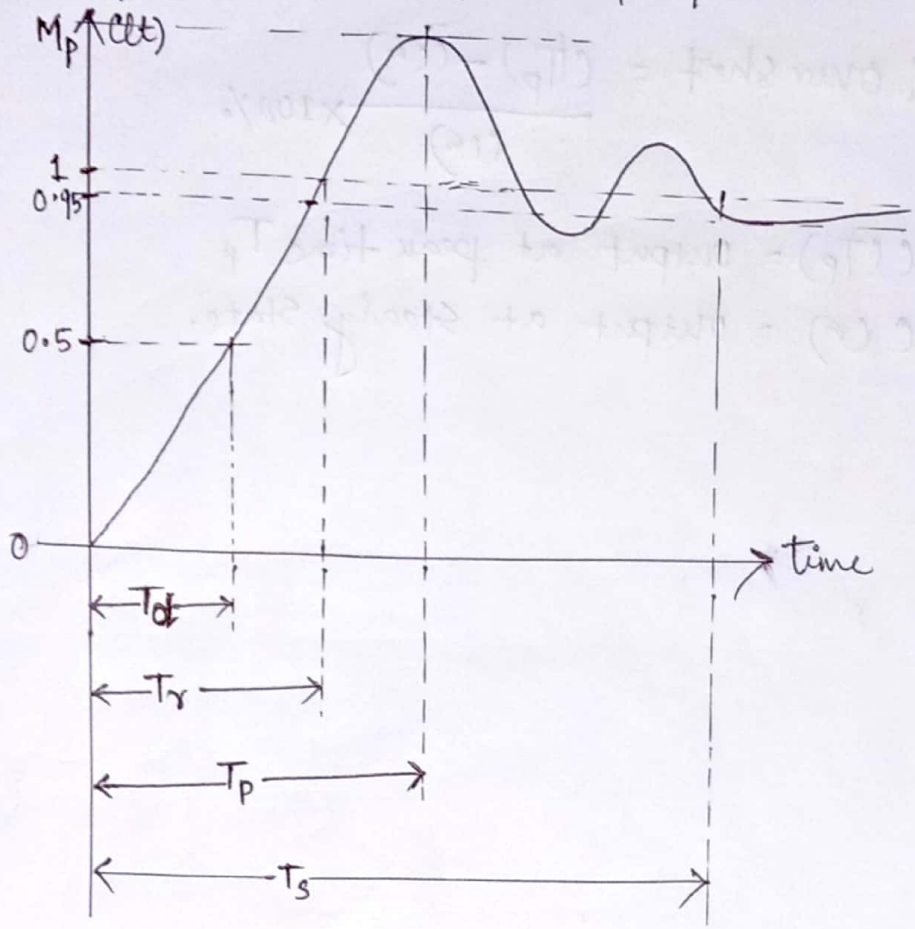
b) An unstable system.



Scanned by CamScanner

Time Response Specifications

Before reaching steady state, all the control system undergoes transient state. this transient state can be analysed in terms of time response specifications.



Delay Time (T_d) :- The time that the system output response takes for the step input, to reach 50% of its final value is called delay time.

Rise Time (T_r) :- The time that the system output response takes for the step input, to reach 100% of its final value is called rise time.

Settling Time (T_s) :- This time is defined as the time required for the step response to settle down and stay within a specified tolerance of the final value.

Scanned by CamScanner

Maximum Overshoot (or) Peak Overshoot :-

The ratio of the maximum value of the step excited output to the final output is called overshoot of the system.

$$\% \text{ overshoot} = \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100\%$$

Where $c(T_p)$ = output at peak time T_p

$c(\infty)$ = output at steady state.

Scanned by CamScanner

Derivation of Time response specifications.

Rise Time (T_r) We will consider the system as underdamped

that means $c(t) = 1$ at $t = T_r$.

we also know for an underdamped system

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

we also know $\omega_d = \omega_n\sqrt{1-\xi^2}$

So $$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t$$

Putting $\sqrt{1-\xi^2} = \sin\phi$.

we can get $\cos\phi = \xi$ and $\tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t\right]$$

Scanned by CamScanner